Invariants of Binary Forms*

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Historical Background by Michael Abramson

This paper is probably the full version of the result announced in P. Gordan's earlier paper *Neuer Beweis des Hilbertschen Satzes über homogene Funktionen [A New Proof of Hilbert's Theorem on Homogeneous Functions]*, Nachr. der Königl. Ges. der Wiss. zu Göttingen 3 (1899), 240-242 (Trans. by M. Abramson, ACM SIGSAM Bulletin 32/2 (1998), 47-48). These two papers contain the earliest known example of a (non-trivial) Gröbner basis in the literature. Section 2.V describes a construction and reduction procedure similar to that of the S-polynomial, including the idea that if the S-polynomial reduces to zero, we have redundancy. Upon continuation of this process, the set N in Section 2.VI is a Gröbner basis, though no proof is given which verifies the procedure actually terminates.

Abstract. The system of binary forms is finite. I proved this theorem and Mr. Hilbert extended it to forms in n variables. In this note, a theorem is given which implies the others.

Chapter 1: Elementary Systems of Products

I. – Systems of Products

The products of n variables

$$x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$$

form a system S if the exponents are subject to the relations

$$(1) \qquad \Theta_1, \Theta_2, \dots .$$

These relations define S.

Example 1. The formula

$$k_1 \equiv 0 \pmod 3$$

defines the system

$$(S) \qquad x_1^3, x_1^6, x_1^9, \dots .$$

Example 2. The formula

$$k_1 + k_2 + k_3 + k_4 \equiv 0 \pmod 3$$

defines the system

$$(S) \qquad x_1^3, x_1^4 x_2^2, x_3^7 x_4^2, \dots .$$

Example 3. The formulae

$$k_1 + k_2 + k_3 + k_4 \equiv 0 \pmod 3$$
$$k_1k_2 + k_1k_3 + k_1k_4 + k_2k_3 + k_2k_4 + k_3k_4 > 0$$

define the system

$$(S) \qquad x_1^2 x_2, x_1 x_2 x_4^4, x_1 x_3^2 x_4^3, \dots .$$

II. – Elementary Sets

It can be shown that the products T in S are divisible by each other. If we omit the products T, which are divisible by other T, we obtain a partial system Σ; it is called the *elementary system* defined by the relations (1). It contains the products

$$(\Sigma) \qquad P_1, P_2, \dots ,$$

which have the following properties:

1. The exponents k satisfy the relations (1).
2. No P is divisible by any other.
3. Each product T of S is divisible by at least one of the P.

* *Les invariants des formes binaires.* J. de Mathématiques Pures et Appliqués **6** (1900), 141-156. Translation by Michael Abramson.

Example 1. The formula

$$k_1 \equiv 0 \qquad (\mathrm{mod}\ 3)$$

defines the elementary system

$$(\Sigma) \qquad x_1^3.$$

Example 2. The formula

$$k_1 + k_2 + k_3 + k_4 \equiv 0 \qquad (\mathrm{mod}\ 3)$$

defines the elementary system

$$(2)(\Sigma) \qquad \begin{cases} x_1^3, x_1^2x_2, x_1^2x_3, x_1^2x_4, x_1x_2^2, \\ x_1x_2x_3, x_1x_2x_4, x_1x_3^2, x_1x_3x_4, x_1x_4^2, \\ x_2^3, x_2^2x_3, x_2^2x_4, x_2x_3^2, x_2x_3x_4, \\ x_2x_4^2, x_3^3, x_3^2x_4, x_3x_4^2, x_4^3 \end{cases}$$

These are the terms of a quaternary cubic form in the usual order. I view each one of them as *complicated* compared to those which follow it and the last ones as *simpler*; thus x_4^3 is the simplest term and x_1^3 is the most complicated.

Example 3. The formulae

$$k_1 + k_2 + k_3 + k_4 \equiv 0 \quad (\mathrm{mod}\ 3)$$
$$k_1k_2 + k_1k_3 + k_1k_4 + k_2k_3 + k_2k_4 + k_3k_4 > 0$$

define the elementary system

$$(3)(\Sigma) \qquad \begin{cases} x_1^2x_2, x_1^2x_3, x_1^2x_4, x_1x_2^2, \\ x_1x_2x_3, x_1x_2x_4, x_1x_3^2, x_1x_3x_4, x_1x_4^2, \\ x_2^2x_3, x_2^2x_4, x_2x_3^2, x_2x_3x_4, \\ x_2x_4^2, x_3^2x_4, x_3x_4^2. \end{cases}$$

III. – Indices of Elementary Sets

The number of products contained in the elementary system Σ is called the *index* of Σ. If the products are composed of n variables, I represent this index by h_n. If we have several elementary systems

$$\Sigma_1, \Sigma_2, \Sigma_3, \dots,$$

I represent their indices by

$$h_{n,1}, h_{n,2}, h_{n,3}, \dots .$$

If $n = 1$, we have

$$(4) \qquad h_1 \leq 1.$$

Example 1. The elementary system defined by the formula

$$k_1 \equiv 0 \qquad (\mathrm{mod}\ 3)$$

has index $h_1 = 1$.

Example 2. The elementary system defined by the formula

$$k_1 + k_2 + k_3 + k_4 \equiv 0 \qquad (\mathrm{mod}\ 3)$$

has index $h_4 = 20$.

Example 3. The elementary system defined by the formulae

$$k_1 + k_2 + k_3 + k_4 \equiv 0 \quad (\mathrm{mod}\ 3)$$
$$k_1k_2 + k_1k_3 + k_1k_4 + k_2k_3 + k_2k_4 + k_3k_4 > 0$$

has index $h_4 = 16$.

IV. – Partial Systems of the Elementary System Σ

If the product

$$P_1 = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$$

is contained in the elementary system Σ, we can form partial systems

L_0 $\qquad P_1,$

L_1 $\qquad x_1^g x_2^{k_2} x_3^{k_3} \cdots x_n^{k_n} = x_1^g Q,$

where $\qquad g \leq \lambda_1;$

L_2 $\qquad x_1^{k_1} x_2^{g-\lambda_1} x_3^{k_3} \cdots x_n^{k_n} = x_2^{g-\lambda_1} Q,$

where $\qquad \lambda_1 < g \leq \lambda_1 + \lambda_2;$

L_3 $\qquad x_1^{k_1} x_2^{k_2} x_3^{g-\lambda_1-\lambda_2} x_4^{k_4} \cdots x_n^{k_n} = x_3^{g-\lambda_1-\lambda_2} Q,$

where $\qquad \lambda_1 + \lambda_2 < g \leq \lambda_1 + \lambda_2 + \lambda_3;$

$\vdots$

L_g $\qquad x_1^{k_1} \cdots x_{n-1}^{k_{n-1}} x_n^{g-\lambda_1-\ldots-\lambda_{n-1}} = x_n^{g-\lambda_1-\ldots-\lambda_{n-1}} Q,$

where $\qquad \lambda_1 + \lambda_2 + \ldots + \lambda_{n-1} < g;$

Their number is

$$1 + \lambda_1 + \lambda_2 + \ldots + \lambda_n = 1 + \rho.$$

Example 1. The elementary system defined by the formula

$$k_1 \equiv 0 \qquad (\mathrm{mod}\ 3)$$

has no partial systems.

Example 2. The elementary system defined by the formula

$$k_1 + k_2 + k_3 + k_4 \equiv 0 \qquad (\text{mod } 3)$$

has partial systems

L_1 $\quad P_1 = x_1^3,$

L_2 $\quad x_1^2x_2, x_1^2x_3, x_1^2x_4,$

L_3 $\quad x_1x_2^2, x_1x_2x_3, x_1x_2x_4, x_1x_3^2, x_1x_3x_4, x_1x_4^2,$

L_4 $\quad x_2^3, x_2^2x_3, x_2^2x_4, x_2x_3^2, x_2x_3x_4, x_2x_4^2, x_3^3, x_3^2x_4, x_3x_4^2, x_4^3.$

Example 3. The elementary system defined by the formulae

$$k_1 + k_2 + k_3 + k_4 \equiv 0 \quad (\text{mod } 3)$$
$$k_1k_2 + k_1k_3 + k_1k_4 + k_2k_3 + k_2k_4 + k_3k_4 > 0$$

has partial systems

L_0 $\quad P_1 = x_1^2x_2,$

L_1 $\quad x_1x_2^2, x_1x_2x_3, x_1x_2x_4, x_1x_3^2, x_1x_3x_4, x_1x_4^2$

L_2 $\quad x_2^2x_3, x_2^2x_4, x_2x_3^2, x_2x_3x_4, x_2x_4^2, x_3^2x_4, x_3x_4^2,$

L_3 $\quad x_1^2x_3, x_1^2x_4, x_1x_3^2, x_1x_3x_4, x_1x_4^2, x_3^2x_4, x_3x_4^2.$

V. – Relations for the Index h_n

If the product

$$P = x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}$$

is contained in Σ and differs from

$$P_1 = x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_n^{\lambda_n},$$

it is not divisible by P_1; it has at least one exponent

$$k_\sigma < \lambda_\sigma$$

and is contained in the partial system L_g, where

$$k_\sigma = g - \lambda_1 - \lambda_2 - \ldots - \lambda_{\sigma-1}.$$

All the P in Σ are contained in the L. If we represent the number of products contained in L_g by l_g we have

(5) $\quad h_n \leq 1 + l_1 + l_2 + l_3 + \ldots + l_\rho.$

Example 1.

(5) $\quad h_1 \leq 1, \quad l_g = 0.$

Example 2.

(5) $\quad h_4 = 20, \quad l_1 = 3, \quad l_2 = 6, \quad l_3 = 10,$

$$20 = 1 + 3 + 6 + 10.$$

Example 3.

(5) $\quad h_1 = 16, \quad l_1 = 6, \quad l_2 = 7, \quad l_3 = 7,$

$$16 < 1 + 6 + 6 + 7.$$

VI. – The Corresponding Systems

The products lying in the partial system L_g are of the form

L_g $\quad x_\sigma^r Q_1, x_\sigma^r Q_2, x_\sigma^r Q_3, \ldots,$

and none of them is divisible by any other. The system

Σ_g $\quad Q_1, Q_2, Q_3, \ldots$

corresponds to the system L_g.

The Q are formed just from the $n-1$ variables

$$x_1, x_2, \ldots, x_{\sigma-1}, x_{\sigma+1}, \ldots, x_n;$$

none of them is divisible by any other. The exponents k of Q are subject to the relations

(1_a) $\quad H_1, H_2, \ldots$

derived from the relations

(1) $\quad \Theta_1, \Theta_2, \ldots.$

The Σ_g are the elementary systems defined by the relations (1_a).

Example 1. As there is no L_g, there is also no corresponding system Σ_g.

Example 2. The elementary systems corresponding to the partial systems

L_1 $\quad x_1^2x_2, x_1^2x_3, x_1^2x_4,$

L_2 $\quad x_1x_2^2, x_1x_2x_3, x_1x_2x_4, x_1x_3^2, x_1x_3x_4, x_1x_4^2,$

L_3 $\quad x_2^3, x_2^2x_3, x_2^2x_4, x_2x_3^2, x_2x_3x_4, x_2x_4^2, x_3^3, x_3^2x_4, x_3x_4^2, x_4^3.$

is

Σ_1 $\quad x_2, x_3, x_4,$

Σ_2 $\quad x_2^2, x_2x_3, x_2x_4, x_3^2, x_3x_4, x_4^2,$

Σ_3 $\quad x_2^3, x_2^2x_3, x_2^2x_4, x_2x_3^2, x_2x_3x_4, x_2x_4^2, x_3^3, x_3^2x_4, x_3x_4^2, x_4^3;$

they are defined by the relations

$$\begin{array}{lll} \Sigma_1 & \text{by} & k_2 + k_3 + k_4 \equiv 1 \quad (\text{mod } 3) \\ \Sigma_2 & \text{by} & k_2 + k_3 + k_4 \equiv 2 \quad (\text{mod } 3) \\ \Sigma_3 & \text{by} & k_2 + k_3 + k_4 \equiv 0 \quad (\text{mod } 3) \end{array}$$

Example 3. The elementary systems corresponding to the partial systems

L_1 $\quad x_1x_2^2, x_1x_2x_3, x_1x_2x_4, x_1x_3^2, x_1x_3x_4, x_1x_4^2,$

L_2 $\quad x_2^2x_3, x_2^2x_4, x_2x_3^2, x_2x_3x_4, x_2x_4^2, x_3^2x_4, x_3x_4^2,$

L_3 $\quad x_1^2x_3, x_1^2x_4, x_1x_3^2, x_1x_3x_4, x_1x_4^2, x_3^2x_4, x_3x_4^2.$

is

$\Sigma_1 \qquad x_2^2, x_2x_3, x_2x_4, x_3^2, x_3x_4, x_4^2,$

$\Sigma_2 \qquad x_2^2x_3, x_2^2x_4, x_2x_3^2, x_2x_3x_4, x_2x_4^2, x_3^2x_4, x_3x_4^2,$

$\Sigma_3 \qquad x_1^2x_3, x_1^2x_4, x_1x_3^2, x_1x_3x_4, x_1x_4^2, x_3^2x_4, x_3x_4^2;$

they are defined by the relations

$$\Sigma_1 \quad \text{by} \quad k_2 + k_3 + k_4 \equiv 2 \pmod 3$$

$$\Sigma_2 \quad \text{by} \quad \begin{cases} k_2 + k_3 + k_4 \equiv 0 & \pmod 3 \\ k_2k_3 + k_2k_4 + k_3k_4 > 0 \end{cases}$$

$$\Sigma_3 \quad \text{by} \quad \begin{cases} k_1 + k_3 + k_4 \equiv 0 & \pmod 3 \\ k_1k_3 + k_1k_4 + k_3k_4 > 0 \end{cases}$$

VII. – Relations Between the Indices of Several Systems

The number of products P contained in L_g is equal to that of the products Q contained in Σ_g

$$l_g = h_{n-1,g}.$$

The formulae corresponding to the formulae (5) are

$$(6) \quad \begin{cases} h_n & \le & 1 + h_{n-1,1} + h_{n-1,2} + \ldots + h_{n-1,\rho}, \\ h_1 & \le & 1, \\ 20 & = & 1 + 3 + 6 + 10, \\ 16 & < & 1 + 6 + 6 + 7. \end{cases}$$

VIII. – The Indices of Elementary Systems are of Finite Number

Proof. $h_1 \le 1$ is a finite number. The numbers

$$(7) \quad \begin{cases} h_2 & \le & 1 + h_{1,1} + h_{1,2} + \ldots + h_{1,\rho_1}, \\ h_3 & \le & 1 + h_{2,1} + h_{2,2} + \ldots + h_{2,\rho_2}, \\ & & \vdots \end{cases}$$

are also finite numbers. □

The numbers ρ are defined §IV by the degrees of whatever products are contained in the Σ.

Chapter 2: Elementary Systems of Homogeneous Functions

I. – The Order of Terms of a Homogeneous Function

The terms of a homogeneous function f of n variables are products

$$P = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}.$$

We assume them to be written in an order such that each one of them precedes those that are simpler. The first term P_1 is the most complicated. If we set

$$f = c_1 P_1 + \chi,$$

the terms of χ are simpler than P_1.

II. – The Order of Homogeneous Functions

We will classify the homogeneous functions

$$f_1, f_2, \ldots$$

according to their degrees, the f of lower degree precede those of higher degree. Forms of the same degree are arranged according to their first terms.

Those whose first terms are simple precede those whose first terms are more complicated.

This order is inverse to the order of terms in the homogeneous function. With the forms f thus ordered, we will view each one of them as *simpler* than those which follow it.

III. – The System S of Homogeneous Functions

The relations

$$\Theta_1, \Theta_2, \ldots$$

which connect the coefficients of the homogeneous functions define a system S consisting of all homogeneous functions

$$(S) \qquad f_1, f_2, \ldots$$

whose coefficients satisfy these relations. The f of (S) are ordered according to §2.II. The systems whose homogeneous functions are simple are called *simple*.

IV. – Derived Systems

Given a system of homogeneous functions

$$(S) \qquad \phi_1, \phi_2, \phi_3, \ldots,$$

we can derive another system

$$(8) \quad \begin{cases} \eta_1 & = & \phi_1 \\ \eta_2 & = & A_{21}\phi_1 + \phi_2 \\ \eta_3 & = & A_{31}\phi_1 + A_{32}\phi_2 + \phi_3 \\ \eta_4 & = & A_{41}\phi_1 + A_{42}\phi_2 + A_{43}\phi_3 + \phi_4 \\ & & \vdots \end{cases}.$$

The A are whatever functions which will make the functions η homogeneous.

V. – Reduction of Derived Systems

It can be shown that among the η of a derived system M

$$\eta_1, \eta_2, \eta_3, \dots,$$

there are two functions (§2.I)

$$\begin{aligned}\eta_\lambda &= c_\lambda P_\lambda + \chi_\lambda,\\ \eta_\mu &= c_\mu P_\mu + \chi_\mu,\end{aligned}$$

such that the first term P_λ of η_λ is divisible by the first term P_μ of η_μ

$$P_\lambda = R P_\mu.$$

In this case, I form the aggregate

$$\overline{\eta_\lambda} = \eta_\lambda - \frac{c_\lambda}{c_\mu} R \eta_\mu = \chi_\lambda - \frac{c_\lambda}{c_\mu} R \chi_\mu, \tag{9}$$

and I substitute it in place of η_λ in the system M.

Let M_1 be the system which results from this substitution.

$\overline{\eta_\lambda}$ and M_1 are simpler than η_λ and M. M is reduced to M_1 by the substitution (9). If we have

$$\overline{\eta_\lambda} = 0, \tag{10}$$

the system M_1 contains one function fewer than M.

In this case, we have, according to (8),

$$0 = A_{\lambda 1}\phi_1 + A_{\lambda 2}\phi_2 + \dots + A_{\lambda,\lambda-1} + \phi_{\lambda-1} + \phi_\lambda, \tag{11}$$

that is to say, ϕ_λ is an aggregate of simpler functions ϕ.

VI. – The Irreducible System N

If we continue the process of reduction, we obtain an irreducible system

$$f_1, f_2, \dots, \tag{N}$$

The first terms of the functions f are the products

$$P_1, P_2, \dots,$$

none of which is divisible by any other. They form an elementary system Σ. Let h be its index. The system N contains h functions

$$f_1, f_2, \dots, f_h. \tag{N}$$

VII. – The Elementary System L

If the number h is less than the number of functions ϕ, we can reduce all but h functions ϕ to simpler functions by formula (11). Represent by Φ those functions which remain. The system

$$\Phi_1, \Phi_2, \dots, \Phi_h \tag{L}$$

is called the *elementary system* defined by the relations

$$\Theta_1, \Theta_2, \dots.$$

VIII. – Hilbert's Theorem

All of the ϕ, except the Φ, are aggregates of the simpler functions ϕ. Continuing this reduction, we obtain the formula

$$\phi = c_1\Phi_1 + c_2\Phi_2 + \dots + c_h\Phi_h. \tag{12}$$

Chapter 3: Application to Invariants

I. – Transformation of Binary Forms

The binary form

$$f = a_0 x_1^n + \binom{n}{1} a_1 x_1^{n-1} x_2 + \binom{n}{2} a_2 x_1^{n-2} x_2^2 + \dots$$

is transformed by the substitution

$$\begin{cases} x_1 = \xi_1 y_1 + \eta_1 y_2 \\ x_1 = \xi_2 y_1 + \eta_2 y_2 \end{cases} \tag{13}$$

into the form

$$f = A_0 y_1^n + \binom{n}{1} A_1 y_1^{n-1} y_2 + \binom{n}{2} A_2 y_1^{n-2} y_2^2 + \dots.$$

The determinant of the substitution is

$$\Delta = \begin{vmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{vmatrix},$$

and the coefficients A are the polars of f with respect to the variables ξ, η.

II. – Series Expansion

Each homogeneous function $F(x, y)$ of two series of cogradient variables

$$x_1, x_2 \qquad \text{and} \qquad y_1, y_2$$

can be expanded as a series whose terms are power products of

$$(xy) = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

and polars of elementary covariants of F. In the symbolic setting, if x and y have the same degree in F,

$$F(x, y) = r_x^m t_y^m,$$

we have the series

$$(14) \qquad F = \sum_k \frac{\binom{m}{k}^2}{\binom{2m-k+1}{k}} (\overline{r, s})_{y^k}^{m-k} (xy)^k.$$

III. – Elementary Systems of Invariants

The invariants i of f are homogeneous integral functions ϕ with coefficients

$$a_0, a_1, a_2, \dots, a_n.$$

The coefficients of ϕ satisfy the relations

$$\Theta_1, \Theta_2, \dots,$$

which are derived from partial differential equations of invariants. Suppose the elementary system L of ϕ is formed from the invariants

$$(L) \qquad I_1, I_2, \dots, I_h.$$

By Hilbert's theorem, each invariant i has the form

$$(15) \qquad i = c_1(a)I_1 + c_2(a)I_2 + \dots + c_h(a)I_h.$$

The functions $c(a)$ are functions of a which make the expression homogeneous.

IV. – Transformation of the Formula (15)

The invariants i of f are obtained by the substitution (13) of powers of Δ into factors. If the invariants

$$i, I_1, I_2, I_3, \dots, I_h$$

have weights

$$\nu, \nu_1, \nu_2, \nu_3, \dots, \nu_h;$$

they obtain the factors

$$\Delta^\nu, \Delta^{\nu_1}, \Delta^{\nu_2}, \Delta^{\nu_3}, \dots, \Delta^{\nu_h}.$$

Formula (15) is transformed by the substitution (13) into

$$(16) \qquad \Delta^\nu i = c_1(A)\Delta^{\nu_1} I_1 + \dots + c_h(A)\Delta^{\nu_h} I_h.$$

V. – Simplification of the Formula (16)

The $c_g(A)$ are functions of polars A and they are covariants of f. In the symbolic setting,

$$(17) \qquad \left\{ \begin{array}{rcl} c_g(A) & = & r_{g,\xi}^{\nu-\nu_g} s_{g,\eta}^{\nu-\nu_g}, \\ \dfrac{1}{\nu-\nu_g} (\overline{r_g, s_g})^{\nu-\nu_g} & = & B_g, \end{array} \right.$$

formula (16) becomes

$$(18) \qquad \Delta^\nu i = r_{1,\xi}^{\nu-\nu_1} s_{1,\eta}^{\nu-\nu_1} \Delta^{\nu_1} I_1 + \dots + r_{h,\xi}^{\nu-\nu_h} s_{h,\eta}^{\nu-\nu_h} \Delta^{\nu_h} I_h.$$

By substituting their series (14) in place of

$$r_{g,\xi}^{\nu-\nu_g} s_{g,\eta}^{\nu-\nu_g},$$

we obtain an identical equation in Δ of degree ν. By comparing coefficients of Δ^ν, we obtain

$$(19) \qquad \Delta^\nu = B_1 i_1 + B_2 i_2 + \dots + B_h i_h.$$

VI. – The Invariants *i* are Integral Functions of *I*

Proof. By arranging the invariants i according to their degrees, invariants of lower degree precede and invariants of higher degree succeed.

By wanting to represent i as an integral function of the I, the invariants B_g are already expressed

$$B_g = F_g(I).$$

If we substitute their values into (18), we obtain

$$i = F_1(I)I_1 + F_2(I)I_2 + \dots + F_h(I)I_h = F(I).$$

□