Computation of Gröbner Bases for Two-Loop Propagator Type Integrals

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The Gröbner basis technique for calculating Feynman diagrams proposed in [1] is applied to the two-loop propagator type integrals with arbitrary masses and momentum. We describe the derivation of Gröbner bases for all integrals with 1PI topologies and present elements of the Gröbner bases.

1. INTRODUCTION

The most efficient algorithms to evaluate Feynman diagrams exploit recursive methods based on integration by parts technique [2] or technique of generalized recurrence relations [3], [4]. It is easy to derive hundreds of recurrence relations but it is by far not so easy to formulate optimal algorithm how to use them for the reduction of a given type of integrals to a minimal set of basis integrals. In [1] it was proposed to use the Gröbner basis method [5] as a mathematical background to solve this problem. This proposal can be realized in different ways. For example, the system of recurrence relations can be rewritten as a system of partial differential equations (PDE). The set of relations needed for reduction of Feynman integrals with different powers of propagators to the set of basis integrals will be Gröbner basis of this overdetermined system of differential equations. Information about the minimal basis of integrals also will be contained in the Gröbner basis. Another possibility will be to rewrite recurrence relations in terms of operators shifting powers of propagators. Each equation should be considered as an operator. Operators shifting powers of propagators satisfy an Ore algebra' condition and therefore, to find minimal set of recurrence relations from our system of equations ammounts to computation of it's Gröbner basis in Ore algebra.

Our preliminary investigation reveals that both approaches have some merits and shortcomings.

The first approach certainly will be preferable in case when integrals depend on many mass scales. For integrals depending on two - three mass scales the second approach can be more efficient.

In this article we present the computation of the Gröbner basis for the two-loop propagator type integrals with arbitrary momentum and masses. The derivation is based on a representation of recurrence relations as partial differential equations. For integrals with three, four and five lines we used notation J_3^d, V^d and F^d respectively which was adopted in our paper [4].

2. TWO-LOOP 'SUNSET' INTEGRALS

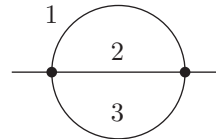We begin our consideration with the so-called, 'sunset' type of integrals given on Fig.1.

Fig.1

Recurrence relations for this type of integrals can be obtained from two equations [6], [2]

$$\int d^d k_1 d^d k_2 \quad \frac{\partial}{\partial k_{1\mu}} \quad \frac{S_{\mu\nu\rho} Q_{\rho\nu}}{c_1^{\nu_1} c_2^{\nu_2} c_3^{\nu_3}} \equiv 0, \qquad (1)$$

$$\int d^d k_1 d^d k_2 \quad \frac{\partial}{\partial k_{2\mu}} \quad \frac{S_{\mu\nu\rho} Q_{\rho\nu}}{c_1^{\nu_1} c_2^{\nu_2} c_3^{\nu_3}} \equiv 0, \qquad (2)$$

where

$$c_1 = k_1^2 - m_1^2, \quad c_2 = (k_1 - k_2)^2 - m_2^2,$$
$$c_3 = (k_2 - q)^2 - m_3^2,$$
$$S_{\mu\nu\rho} = w_1 g_{\mu\nu} q_\rho +$$
$$w_2 g_{\rho\nu} q_\mu + w_3 g_{\mu\rho} q_\nu + w_4 q_\mu q_\nu q_\rho,$$

and $Q_{\mu\nu}$ is an arbitrary tensor. In our consideration we restrict ourselves to the second rank tensor in k_1, k_2 and external momentum q:

$$Q_{\mu\nu} = x_1 k_{1\nu} k_{1\mu} + x_2 k_{1\nu} k_{2\mu} + x_3 k_{1\mu} k_{2\nu}$$
$$+ x_4 k_{2\nu} k_{2\mu} + x_5 k_{1\nu} q_\mu + x_6 k_{1\mu} q_\nu \quad (3)$$
$$+ x_7 k_{2\nu} q_\mu + x_8 k_{2\mu} q_\nu + x_9 q_\mu q_\nu.$$

Parameters x_i, w_j are arbitrary constants. After differentiation integrals with irreducible numerators

$$\int \int d^d k_1 d^d k_2 \frac{(k_2 q)^\alpha (k_1 k_2)^\beta}{c_1^{\nu_1} c_2^{\nu_2} c_3^{\nu_3}}$$

were expressed in terms of integrals with shifted dimension. For example,

$$\int \int d^d k_1 d^d k_2 \frac{(k_1 q)(k_2 q)}{c_1 c_2 c_3} =$$
$$\frac{q^2}{2\pi^2} \int \int \frac{d^{d+2} k_1 d^{d+2} k_2}{c_1 c_2^2 c_3} \quad (4)$$
$$+ \frac{q^4}{\pi^4} \int \int \frac{d^{d+4} k_1 d^{d+4} k_2}{c_1 c_2^2 c_3^3} \left[\frac{2}{c_1} + \frac{4}{c_2} \right].$$

Propagators with 'shifted' indices should be represented as:

$$\frac{1}{c_j^{1+r}} = \frac{1}{r!} \frac{\partial^r}{\partial z_j^r} \frac{1}{c_j}, \quad (5)$$

where

$$z_j = m_j^2.$$

'Sunset' type integrals J_3^d with different shifts of the parameter of the space-time dimension d were considered as different functions, i.e:

$$J_3^d = \text{sun0}, \quad J_3^{d+2} = \text{sun1}, \quad J_3^{d+4} = \text{sun2}, \ldots \quad (6)$$

These substitutions allows one to transform the system of recurrence relations into a linear system of PDEs for the vector function $P \equiv \{sun0, sun1, \ldots\}$. At $\nu_1 = \nu_2 = \nu_3 = 1$ Eqs.

(1),(2) give 28 different relations connecting 'sunset' integrals with different shifts of the parameter of space-time dimension d and different products of one-loop tadpole type integrals.

To find the Gröbner basis of our system of PDE we used package Rif [7] which is distributed as part of Maple.

Just for illustration we give a concrete example of the program written in Maple for computation of the Gröbner basis.

```
read 'equations':
eq29:=z1*diff(T1(z1),z1)
                - (d/2-1)*T1(z1)=0:
eq30:=z2*diff(T2(z2),z2)
                - (d/2-1)*T2(z2)=0:
eq31:=z3*diff(T3(z3),z3)
                - (d/2-1)*T3(z3)=0:
syst:={seq(eq||j,j=1..31)}:
syst:= syst union {eq29,eq30,eq31}:
syst:= convert(syst,list):

with(DEtools):

invr := [z1,z2,z3]:
dvars:=[sun3,sun2,sun1,sun0,T1,T2,T3]:
Rnk:=[[0,0,0,150,100,50,0,0,0,0],
      [1,1,1,  0,  0, 0,0,0,0,0]]:
bas := rifsimp(
  syst,dvars,indep=invr,ranking=Rnk);
GBasis:= bas[Solved];
quit;
```

First, the system of 28 differential equations stored in the file 'equations' is read in and then we add to it three differential relations for one-loop tadpole integrals with different masses:

$$\frac{\partial}{\partial z_j} T_j(z_j) = \frac{d-2}{2z_j} T_j(z_j), \quad T_j(z_j) = \int \frac{d^d k_1}{k_1^2 - z_j}.$$

In order to get rid of integrals with shifts of d we adopted here a ranking giving weight proportional to the shift of space time dimension. Also derivatives of functions have weight higher than the function itself.

It took 5 min on a 1.6 GHz PC to obtain the Gröbner basis consisting of 19 differential relations for J_3^d with different shifts of d and three

relations for $T_i(z_i)$. The left hand sides of these 19 relations are:

$$\frac{\partial^2 J_3^d}{\partial z_1{}^2}, \quad \frac{\partial^2 J_3^d}{\partial z_1 \partial z_2}, \quad \frac{\partial^2 J_3^d}{\partial z_1 \partial z_3}, \quad \frac{\partial^2 J_3^d}{\partial z_2{}^2},$$

$$\frac{\partial^2 J_3^d}{\partial z_2 \partial z_3}, \quad \frac{\partial^2 J_3^d}{\partial z_3{}^2},$$

$$\frac{\partial^2 J_3^{d+2}}{\partial z_1 \partial z_2}, \quad \frac{\partial^2 J_3^{d+2}}{\partial z_1 \partial z_3}, \quad \frac{\partial^2 J_3^{d+2}}{\partial z_2 \partial z_3},$$

$$\frac{\partial^5 J_3^{d+4}}{\partial z_3{}^2 \partial z_1{}^3}, \quad \frac{\partial^5 J_3^{d+4}}{\partial z_1{}^2 \partial z_3{}^3}, \quad \frac{\partial^5 J_3^{d+4}}{\partial z_1 \partial z_2 \partial z_3{}^3},$$

$$\frac{\partial^5 J_3^{d+4}}{\partial z_3 \partial z_2{}^3 \partial z_3}, \quad \frac{\partial^5 J_3^{d+4}}{\partial z_3{}^2 \partial z_2{}^2 \partial z_3}, \quad \frac{\partial^4 J_3^{d+4}}{\partial z_1{}^2 \partial z_2 \partial z_3},$$

$$\frac{\partial^4 J_3^{d+4}}{\partial z_1 \partial z_2{}^2 \partial z_3}, \quad \frac{\partial^7 J_3^{d+6}}{\partial z_1{}^2 \partial z_2{}^2 \partial z_3{}^3},$$

$$\frac{\partial^7 J_3^{d+6}}{\partial z_1{}^3 \partial z_2 \partial z_3{}^3}, \quad \frac{\partial^7 J_3^{d+6}}{\partial z_1 \partial z_2{}^3 \partial z_3{}^3}. \tag{7}$$

In complete agreement with [4] the number of equations for J_3^d without shifts of d is enough to reduce integrals with $\nu_1 + \nu_2 + \nu_3 > 4$ to four basis integrals $J_3^d, \partial J_3^d/\partial z_1, \partial J_3^d/\partial z_2, \partial J_3^d/\partial z_3$, and three products of one-loop tadpoles. The basis for integrals with shifted d is not complete. As was shown in [4] these integrals can be expressed in terms of J_3^d and its first derivatives. Such a relation was not obtained from the 28 equations produced by Eqs. (1)-(2). It turns out that if we shift $d \rightarrow d+2$ in (1),(2) and add the obtained 28 equations to original 28 equations then from this enlarged system we will obtain in the Gröbner basis the required expression for J_3^{d+2} which agrees with [4]. Therefore, computing Gröbner basis we must include these equations in our system from the very beginning.

Taking into account the relation for J_3^{d+2} in terms of $J_3^d, \partial J_3^d/\partial z_1, \partial J_3^d/\partial z_2, \partial J_3^d/\partial z_3$, we will find that the Gröbner basis for 'sunset' integrals consists out of 10 relations: 6 relations with second derivatives for J_3^d, one relation for J_3^{d+2} and three relations for one loop tadpole integrals. Expressions for derivatives of $J_3^{d+2}, J_3^{d+4}, J_3^{d+6}$ will be excluded from (7).

It should be noted that keeping in Q only linear terms in k_1, k_2 gives incomplete basis for J_3^d.

By using the Gröbner basis reduction of J_3^d, $J_3^{d+2} \ldots$ integrals with $\nu_1 + \nu_2 + \nu_3 > 4$ can be done with elimination commands built in Maple. We wrote our own procedure for such a reduction taking into account specific properties of our Gröbner basis. This procedure works essentially faster than built in tools of Maple.

3. INTEGRALS V^d

In the same manner as it was done for the 'sunset' integrals we repeated computation of the Gröbner basis for propagator type integrals with the topology given in Fig.2.
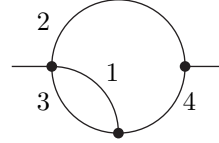

Fig. 2.

The Gröbner basis consists of ten relations which were already obtained for 'sunset' integrals, five relations for V^d and three relations for 1-loop propagator integrals

$$G^d = \int \frac{d^d k_1}{[k_1^2 - z_2][(k_1 - q)^2 - z_4]}. \tag{8}$$

New relations needed in the Gröbner basis are:

$$\frac{\partial V^d}{\partial z_1}, \quad \frac{\partial V^d}{\partial z_2}, \quad \frac{\partial V^d}{\partial z_3}, \quad \frac{\partial V^d}{\partial z_4}, \quad V^{d+2},$$

$$\frac{\partial G^d}{\partial z_2}, \quad \frac{\partial G^d}{\partial z_4}, \quad G^{d+2}.$$

As it was for the 'sunset' case the number of elements in the Gröbner basis and the number of basic integrals for V type integrals is in agreement with [4].

4. INTEGRALS F^d

Gröbner basis for F^d integrals with the topology given in Fig.3 includes basis of 'sunset' type integrals, V type integrals and also new relations.
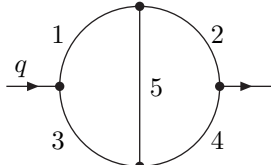
Fig.3.

Six new relations for the following quantities

$$\frac{\partial F^d}{\partial z_1}, \quad \frac{\partial F^d}{\partial z_2}, \quad \frac{\partial F^d}{\partial z_3}, \quad \frac{\partial F^d}{\partial z_4}, \quad \frac{\partial F^d}{\partial z_5}, \quad F^{d+2}, \quad (9)$$

appeared in the Gröbner basis of F^d integrals. Again these relations and the number of master integrals agree with the results given in [4].

5. CONCLUDING REMARKS

The considered examples demonstrate that with already existing software one can compute Gröbner bases for a rather complicated type of Feynman integrals. Our preliminary investigation reveals also that Gröbner bases for vertex integrals with arbitrary masses and external momenta can be computed with the existing computers and available software. Corresponding expressions are rather long, sometimes thousands lines of codes. However after the Gröbner basis is computed, it can be stored and used in different applications. At the present time we cannot use Gröbner bases for reduction of integrals with arbitrary kinematics. The reason is that some masses or Gram determinants appear in denominators and therefore one cannot make reductions when these factors are zero. Different solutions of this problem are possible and they will be presented in future publications.

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