Available online at www.sciencedirect.com
SCIENCE DIRECT®

Journal of Symbolic Computation 40 (2005) 1039–1052

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

Gröbner bases and syzygies on bimodules over PBW algebras☆

Manuel García Román, Socorro García Román*

Departamento de Matemática Fundamental, Universidad de La Laguna, E-38271 La Laguna, Spain

Received 28 May 2004; accepted 5 May 2005
Available online 17 June 2005

Abstract

A new method for the computation of two-sided Gröbner bases of ideals and bimodules shifting the problem to the enveloping algebra is proposed. This alternative method appears to be more efficient than the one in [Kandri-Rody, A., Weispfenning, V., 1990. Non-commutative Gröbner bases in algebras of solvable type. J. Symbolic Comput. 9, 1–26] since it calls the left Buchberger algorithm once. We introduce the notion, arising from the ideals that this method involves, of two-sided syzygy, which is revealed to be useful in the computation of, e.g., the intersection of bimodules. Further applications are left for a later work.

Keywords: Bimodules; Gröbner bases; Syzygy bimodule

1. Introduction

Though first developed in the ring of polynomials, the methods based on Gröbner bases also work in some non-commutative rings, e.g. the Weyl algebras or, more generally, the so-called Poincaré–Birkhoff–Witt rings (PBW, for short), including some classical quantum groups. After first results were obtained in the Weyl algebra (Galligo, 1982) and in tensor algebras of finite-dimensional Lie algebras (Apel and Lassner, 1985), Kandri-Rody

☆ Research supported by PI2002/075, Gobierno de Canarias.
* Corresponding author. Tel.: +34 922 318 168; fax: +34 922 318 145.
E-mail address: sgarcia@ull.es (S.G. Román).

0747-7171/$ - see front matter
doi:10.1016/j.jsc.2005.05.002

and Weispfenning were the first to introduce Gröbner bases in the more general class of algebras having a PBW basis where the degree of a skew-commutator $p_{ij} = x_j x_i - c_{ij} x_i x_j$ is bounded by the degree of the product of generators $x_i x_j$, for $1 \leq i < j \leq n$ (see Kandri-Rody and Weispfenning, 1990). These algebras, so-called solvable polynomial algebras, may also be found under the names of PBW algebras (see Bueso et al., 2003) and G-algebras (see Levandovskyy and Schönemann, 2003). The theory has been surveyed in Kredel (1993) and, recently, in Bueso et al. (1998) and Li (2002). Algorithms for computing the Gelfand–Kirillov dimension, for checking whether a two-sided ideal is prime or not and for computing the projective dimension of a module have also been developed (see Bueso et al., 1996; Lobillo, 1998; Bueso et al., 1999; Gago-Vargas, 2003).

In these generalizations, the authors were mainly interested in one-sided ideals and modules, whereas methods for the two-sided counterparts are adaptations constructed in order to cope with the two-sided input data (cf. Pesch, 1998; Bueso et al., 2003).

In this note we show that those "mends" are not necessary, due to the very well known fact that two-sided ideals and bimodules may be seen as left modules on the enveloping algebra. First, we show that the enveloping algebra of a PBW algebra is another PBW algebra. Second, we find a method for shifting the data back and forth through the morphism

$$\mathfrak{m}^s : (R^{\mathrm{env}})^s \longrightarrow R^s; \quad (f_i \otimes g_i)_{i=1}^s \longmapsto (f_i g_i)_{i=1}^s$$

in order to carry out the computations on the enveloping algebra using one-sided techniques.

This philosophy allows us, for example, to compute Gröbner bases for bimodules with only one call to the left Buchberger algorithm, instead of the a priori unknown number of calls typical of the aforementioned methods (cf. García Román and García Román, 2001).

The techniques that we use have led us to study the syzygy bimodule, which is the two-sided counterpart of the left syzygy module. We show that, amongst its applications, it can be used in the computation of intersections of bimodules when one starts, as usual, from two-sided input data.

Throughout this paper we will use the following notation. We will denote by ϵ_i the element $(0, \ldots, \overset{i}{1}, \ldots, 0) \in \mathbb{N}^n$. The symbol x^α will denote the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in the free algebra $k\langle x_1, \ldots, x_n\rangle$ or in any of its epimorphic images, where k is a field. R will be a PBW k-algebra, R^{op} its opposite algebra and R^{env} its enveloping algebra $R \otimes_k R^{\mathrm{op}}$. Furthermore, for any subset F of the free left R-module R^s, we will denote by ${}_R\langle F\rangle$ (resp. ${}_R\langle F\rangle_R$) the left R-module (resp. the R-bimodule) generated by F. If $\{x^\alpha \,/\, \alpha \in \mathbb{N}^n\}$ is a k-basis of the k-algebra R, then for any $(\alpha, i) \in \mathbb{N}^{n,(s)} = \mathbb{N}^n \times \{1, \ldots, s\}$, we will denote as $\boldsymbol{x}^{(\boldsymbol{\alpha},\boldsymbol{i})}$ the element $(0, \ldots, \overset{i}{x^\alpha}, \ldots, 0)$ of the free R-bimodule R^s. Finally, if $\boldsymbol{f} \in R^s \setminus \{0\}$ is such that $\boldsymbol{f} = \sum c_{(\gamma,j)} \boldsymbol{x}^{(\boldsymbol{\gamma},\boldsymbol{j})}$, then $\exp(\boldsymbol{f})$ (or $\exp_{R^s}(\boldsymbol{f})$, if we want to stress that $\boldsymbol{f}$ is in R^s) will denote $(\alpha, i) = \max\{(\gamma, j);\ c_{(\gamma,j)} \neq 0\}$ relative to a given order in $\mathbb{N}^{n,(s)}$. In that case, i will be called the level of $\boldsymbol{f}$.

The computations of the examples shown in this paper were done using a library of procedures built by the authors using the package of symbolic computation Maple 6. Also the old two-sided Gröbner bases algorithm (as it appears in Kandri-Rody and Weispfenning (1990) or in Bueso et al. (2003)) was coded in this library in order to compare the outputs and the computation times. The computation times correspond to a Pentium III 700 MHz personal computer with 192 MB RAM.

2. The enveloping algebra of a PBW algebra

A subset $Q = \{x_j x_i - q_{ji} x_i x_j - p_{ji};\ 1 \le i < j \le n\}$ of the free algebra $k\langle x_1, \ldots, x_n\rangle$ is a set of quantum relations bounded by the admissible order "$\preceq$" on $\mathbb{N}^n$ if $q_{ji} \in k^*$ and p_{ji} is a finite k-linear combination of standard monomials $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ ($\alpha \in \mathbb{N}^n$) such that $\exp(p_{ji}) \prec \epsilon_i + \epsilon_j$, for all $i < j$, where $\exp(f)$ denotes the exponent of the leading term of the finite k-linear combination of standard monomials f. Following Kandri-Rody and Weispfenning (1990) or Bueso et al. (1998), a *Poincaré–Birkhoff–Witt algebra* (PBW algebra, for short) is a k-algebra R such that the set of standard monomials $\{x^\alpha;\ \alpha \in \mathbb{N}^n\}$ is a k-basis and there exists a set of quantum relations Q bounded by an admissible order "$\preceq$" satisfying

$$R = \frac{k\langle x_1, \ldots, x_n\rangle}{{}_{k\langle x_1,\ldots,x_n\rangle}\langle\ Q\ \rangle_{k\langle x_1,\ldots,x_n\rangle}}.$$

This algebra is usually denoted by $k\{x_1, \ldots, x_n;\ Q, \preceq\}$ (cf. Kredel, 1993).

As proved in Levandovskyy (2003), given a set of bounded quantum relations $Q = \{x_j x_i - q_{ji} x_i x_j - p_{ji};\ 1 \le i < j \le n\}$, the set of standard monomials is a PBW basis of R as above if and only if, for all $1 \le i < j < k \le n$, the element

$$q_{ki} q_{kj} p_{ji} x_k - x_k p_{ji} + q_{kj} x_j p_{ki} - q_{ji} p_{ki} x_j + p_{kj} x_i - q_{ji} q_{ki} x_i p_{kj}$$

is reduced to 0 by Q in $k\langle x_1, \ldots, x_n\rangle$. This property, known as the *non-degeneracy condition*, may be checked by a computer. It corresponds to the overlap ambiguities of Bergman for being resolvable (see Bergman, 1978) or the (noetherian) rewriting system arising from Q for being complete (see Kobayashi, 2004).

Amongst the examples of PBW algebras, we find the commutative polynomial ring $k[x_1, \ldots, x_n]$, some iterated Ore extensions such as the Weyl algebra $A_n(k)$, the enveloping algebra of any finite dimensional Lie algebra, a fairly large class of quantum groups just like the multiparameter n-dimensional quantum space $\mathcal{O}_q(\mathbb{A}^n)$ or the bialgebra of quantum matrices $M_q(2)$, etc.

The tensor product of PBW algebras is a new PBW algebra:

Proposition 1. *If $R = k\{x_1, \ldots, x_m;\ Q_R, \preceq_R\}$ and $S = k\{y_1, \ldots, y_n;\ Q_S, \preceq_S\}$ are PBW algebras with quantum relations*

$$Q_R = \{x_j x_i - q_{ji} x_i x_j - p_{ji};\ 1 \le i < j \le m\},$$
$$Q_S = \{y_j y_i - q'_{ji} y_i y_j - p'_{ji};\ 1 \le i < j \le n\},$$

then $R \otimes_k S$ is the PBW algebra denoted by

$$k\{x_1 \otimes 1, \ldots, x_m \otimes 1, 1 \otimes y_1, \ldots, 1 \otimes y_n;\ Q, \preceq\}, \quad \text{where}$$

$$Q = \left\{ \begin{array}{lr} (x_j \otimes 1)(x_i \otimes 1) - q_{ji}(x_i \otimes 1)(x_j \otimes 1) - p_{ji} \otimes 1; & 1 \leq i < j \leq m \\ (1 \otimes y_j)(x_i \otimes 1) - (x_i \otimes 1)(1 \otimes y_j); & 1 \leq i \leq m,\ 1 \leq j \leq n \\ (1 \otimes y_j)(1 \otimes y_i) - q'_{ji}(1 \otimes y_i)(1 \otimes y_j) - 1 \otimes p'_{ji}; & 1 \leq i < j \leq n. \end{array} \right\}$$

and "$\preceq$" is one amongst the elimination orders (see Adams and Loustaunau, 1994, page 69 for the definition) arising from "$\preceq_R$" and "$\preceq_S$".

Note that if $f \in R \setminus \{0\}$ and $g \in S \setminus \{0\}$, then $\exp(f \otimes g) = (\exp(f), \exp(g)) \in \mathbb{N}^{m+n}$, where $\exp(f \otimes g)$ is computed using one of the elimination orders as above.

As a first example, note that $A_{n+m}(k)$ is the PBW algebra $A_n(k) \otimes A_m(k)$ constructed in the proposition. Another example of this construction is the enveloping algebra R^{env} of $R = k\{x_1, \ldots, x_n;\ Q, \preceq\}$. Before we describe it, let us define the composition orders. For any $\alpha = (\alpha_1, \ldots, \alpha_1) \in \mathbb{N}^n$, denote by α^{op} the n-tuple $(\alpha_n, \ldots, \alpha_1)$.

Definition 2. Let "$\preceq$" be an order on $\mathbb{N}^n$. The *up-component* composition order in $\mathbb{N}^{2n}$, denoted "$\preceq^c$", is defined by

$$(\alpha, \beta) \prec^c (\gamma, \delta) \Leftrightarrow \begin{cases} \alpha + \beta^{\text{op}} \prec \gamma + \delta^{\text{op}}, \text{ or} \\ \alpha + \beta^{\text{op}} = \gamma + \delta^{\text{op}} \text{ and } \beta^{\text{op}} \prec \delta^{\text{op}}. \end{cases}$$

The *down-component* composition order "$\preceq_c$" is defined by

$$(\alpha, \beta) \prec_c (\gamma, \delta) \Leftrightarrow \begin{cases} \alpha + \beta^{\text{op}} \prec \gamma + \delta^{\text{op}}, \text{ or} \\ \alpha + \beta^{\text{op}} = \gamma + \delta^{\text{op}} \text{ and } \alpha \prec \gamma. \end{cases}$$

If "$\preceq$" is an admissible order on $\mathbb{N}^n$, then both composition orders "$\preceq^c$" and "$\preceq_c$"are admissible orders on $\mathbb{N}^{2n}$.

The *write oppositely* morphism is the k-automorphism $\cdot^{op} : k\langle x_1, \ldots, x_n\rangle \longrightarrow k\langle x_1, \ldots, x_n\rangle$ given by $(x_{i_1} \cdots x_{i_n})^{op} = x_{i_n} \cdots x_{i_1}$ for $i_j \in \{1, \ldots, n\}$. Note that the opposite algebra R^{op} is the PBW algebra $k\{x_n, \ldots, x_1;\ Q^{\text{op}}, \preceq^{\text{op}}\}$, where the elements of Q^{op} are those of Q written oppositely and "$\preceq^{\text{op}}$" is the order in $\mathbb{N}^n$ given by $\alpha \preceq^{\text{op}} \beta \iff \alpha^{\text{op}} \preceq \beta^{\text{op}}$. Indeed, the set $\{x_n^{\alpha_n} \cdots x_1^{\alpha_1}\}_{\alpha \in \mathbb{N}^n}$ is a k-basis of R^{op}, and $\exp(p_{ji}^{op}) \prec^{\text{op}} \epsilon_{n-i+1} + \epsilon_{n-j+1}$ for $n \geq j > i \geq 1$.

Proposition 3. *If $R = k\{x_1, \ldots, x_n;\ Q, \preceq\}$ is a PBW algebra with quantum relations $Q = \{x_j x_i - q_{ji} x_i x_j - p_{ji};\ 1 \leq i < j \leq n\}$, then R^{env} is the PBW algebra $k\{x_1 \otimes 1, \ldots, x_n \otimes 1, 1 \otimes x_n, \ldots, 1 \otimes x_1;\ Q^*, \preccurlyeq\}$, where*

$$Q^* = \left\{ \begin{array}{lr} (x_j \otimes 1)(x_i \otimes 1) - q_{ji}(x_i \otimes 1)(x_j \otimes 1) - p_{ji} \otimes 1; & 1 \leq i < j \leq n \\ (1 \otimes x_j)(x_i \otimes 1) - (x_i \otimes 1)(1 \otimes x_j); & 1 \leq i, j \leq n \\ (1 \otimes x_i)(1 \otimes x_j) - q_{ji}(1 \otimes x_j)(1 \otimes x_i) - 1 \otimes p_{ji}^{op}; & 1 \leq i < j \leq n. \end{array} \right\}$$

and "$\preccurlyeq$" is, either any of the elimination orders "$\preceq^*$" or "$\preceq_*$" in $\mathbb{N}^{2n}$ corresponding to "$\preceq$" and "$\preceq^{\text{op}}$", or any of the composition orders "$\preceq^c$" or "$\preceq_c$" on $\mathbb{N}^{2n}$ corresponding to "$\preceq$".

If $f \in R \setminus \{0\}$ and $g \in R^{\text{op}} \setminus \{0\}$, then $\exp(f \otimes g) = (\exp(f), \exp(g)) \in \mathbb{N}^{2n}$ not only for the elimination orders but also if any of the composition orders is considered in R^{env}.

In what follows, we will work on the free R-bimodule R^s, where s is a positive integer and R is a PBW algebra, and we will use the R-module basis $\{\boldsymbol{e_i}\}_{i=1}^s$ consisting of $\boldsymbol{e_i} = (0, \ldots, \overset{i}{1}, \ldots, 0) \in R^s$ for all $1 \le i \le s$.

The notion and some applications of left Gröbner bases in PBW algebras and left modules may be found, e.g., in Bueso et al. (2003). For convenience, just recall that if $M \subset R^s$ is an R-bimodule, then $G = \{\boldsymbol{g_1}, \ldots, \boldsymbol{g_r}\} \subset M \setminus \{0\}$ is a two-sided Gröbner basis for M if one of the following equivalent statements holds:

(1) $M = {}_R\langle G\rangle_R$ and $\mathrm{Exp}\,(M) = \bigcup_{k=1}^r(\mathbb{N}^n + \exp\,(\boldsymbol{g_k}))$;
(2) G is a left Gröbner basis and $M = {}_R\langle G\rangle_R = {}_R\langle G\rangle$;
(3) G is a left Gröbner basis, $M = {}_R\langle G\rangle_R$ and $\boldsymbol{g_k}x_i \in {}_R\langle G\rangle$, for all $k \in \{1, \ldots, r\}$ and $i \in \{1, \ldots, n\}$.

A set $G \subset R^s$ is said to be a *two-sided Gröbner basis* if G is a two-sided Gröbner basis for the R-bimodule ${}_R\langle G\rangle_R$.

3. Computing two-sided Gröbner bases

This section is devoted to the methods for computation of two-sided Gröbner bases for R-subbimodules of R^s. We denote by $\boldsymbol{f} \otimes \boldsymbol{g}$ the element $(f_1 \otimes g_1, \ldots, f_s \otimes g_s) \in (R^{\mathrm{env}})^s$, where $\boldsymbol{f} = (f_1, \ldots, f_s)$, $\boldsymbol{g} = (g_1, \ldots, g_s) \in R^s$.

As a consequence of the third characterization of two-sided Gröbner bases above, some authors have proposed an algorithm for computing them (see Kandri-Rody and Weispfenning, 1990; Bueso et al., 2003). Alternatively, we propose a new algorithm which improves on that one, since it calls the left Buchberger algorithm only once, although it uses more variables and input elements. The philosophy is to transform the problem into computing a left Gröbner basis in the free module $(R^{\mathrm{env}})^s$. This may be done since, just as we saw in the previous section, R^{env} has a PBW structure.

It is known that R-bimodules are exactly left R^{env}-modules. Note that, in particular, the free module R^s is a left R^{env}-module with the action $(r \otimes r')\,\boldsymbol{f} = (rf_1r', \ldots, rf_sr')$, and $(R^{\mathrm{env}})^s$ possesses an R-bimodule structure whose multiplications are given by $r(\boldsymbol{f} \otimes \boldsymbol{g})r' = (rf_1 \otimes g_1r', \ldots, rf_s \otimes g_sr')$, where $r, r' \in R$ and $\boldsymbol{f} = (f_1, \ldots, f_s)$, $\boldsymbol{g} = (g_1, \ldots, g_s) \in R^s$.

Likewise, the map $\mathfrak{m}^s = \mathfrak{m} \times \overset{s}{\cdots} \times \mathfrak{m} : (R^{\mathrm{env}})^s \to R^s$, where $\mathfrak{m}(r \otimes r') = rr'$, for $r, r' \in R$, is an epimorphism of left R^{env}-modules. Thus there exists a bijection

$$\begin{array}{rcl} \{N \subseteq (R^{\mathrm{env}})^s;\ \mathrm{Ker}(\mathfrak{m}^s) \subseteq N \in R^{\mathrm{env}}\!-\!\mathrm{Mod}\} & \longrightarrow & \{M \subseteq R^s;\ M \in R\!-\!\mathrm{Bimod}\} \\ N & \to & M_N := \mathfrak{m}^s(N), \\ N_M := (\mathfrak{m}^s)^{-1}(M) & \leftarrow & M. \end{array}$$

Using this bijection, for each R-bimodule $M \subset R^s$, we have a left R^{env}-module $N_M \subset (R^{\mathrm{env}})^s$. Moreover, from a finite generator system for M it is possible to obtain one for N_M, just as the following results show.

Lemma 4. *Let R be a k-algebra.*

(1) *If $M = {}_R\langle \boldsymbol{f_1}, \ldots, \boldsymbol{f_t}\rangle_R \in R^s$, then $N_M = {}_{R^{\mathrm{env}}}\langle \boldsymbol{f_1} \otimes \boldsymbol{1}, \ldots, \boldsymbol{f_t} \otimes \boldsymbol{1}\rangle + \mathrm{Ker}(\mathfrak{m}^s)$.*

(2) $\operatorname{Ker}(\mathfrak{m}^s) = {}_{R^{\mathrm{env}}}\langle \boldsymbol{f} \otimes \mathbf{1} - \mathbf{1} \otimes \boldsymbol{f};\ \boldsymbol{f} \in R^s \rangle$.

(3) *If* $R = k\{x_1, \ldots, x_n;\ Q;\ \preceq\}$ *is a PBW algebra, then* ${}_{R^{\mathrm{env}}}\langle \boldsymbol{f} \otimes \mathbf{1} - \mathbf{1} \otimes \boldsymbol{f};\ \boldsymbol{f} \in R^s \rangle = {}_{R^{\mathrm{env}}}\langle \boldsymbol{x}^{(\epsilon_j,k)} \otimes \mathbf{1} - \mathbf{1} \otimes \boldsymbol{x}^{(\epsilon_j,k)};\ 1 \le j \le n, 1 \le k \le s\rangle$.

Corollary 5. *If* $R = k\{x_1, \ldots, x_n;\ Q;\ \preceq\}$ *is a PBW algebra and* $M = {}_R\langle \boldsymbol{f_1}, \ldots, \boldsymbol{f_t}\rangle_R \subseteq R^s$*, then*

$$N_M = {}_{R^{\mathrm{env}}}\langle\ \{\boldsymbol{f_i} \otimes \mathbf{1}\}_{i=1}^t, \{\boldsymbol{x}^{(\epsilon_j,k)} \otimes \mathbf{1} - \mathbf{1} \otimes \boldsymbol{x}^{(\epsilon_j,k)}\}_{1\le j\le n,\ 1\le k\le s}\ \rangle.$$

Given an admissible order in $\mathbb{N}^n$, from here on we will name as TOP (term over position) [resp. POT (position over term)] the orders in $\mathbb{N}^{n,(s)}$ given by

$$(\alpha, i) \prec (\beta, j) \iff \begin{cases} \alpha \prec \beta, \text{ or} \\ \alpha = \beta \text{ and } i > j \end{cases} \quad \text{resp.} \begin{cases} i > j, \text{ or} \\ i = j \text{ and } \alpha \prec \beta. \end{cases}$$

Lemma 6. *Let* $R = k\{x_1, \ldots, x_n;\ Q;\ \preceq\}$ *be a PBW algebra and consider the order TOP (or POT) on both* R^s *and* $(R^{\mathrm{env}})^s$*.*

- *Taking* $\preceq^*$ *or* $\preceq^c$ *on* R^{env}*, if* $\boldsymbol{h} \in (R^{\mathrm{env}})^s$ *is such that* $\exp_{(R^{\mathrm{env}})^s}(\boldsymbol{h}) = ((\alpha, 0), i) \in \mathbb{N}^{2n,(s)}$*, then* $\boldsymbol{h} \notin \operatorname{Ker}(\mathfrak{m}^s)$ *and* $\exp_{R^s}(\mathfrak{m}^s(\boldsymbol{h})) = (\alpha, i)$.
- *Taking* $\preceq_*$ *or* $\preceq_c$ *on* R^{env}*, if* $\boldsymbol{h} \in (R^{\mathrm{env}})^s$ *is such that* $\exp_{(R^{\mathrm{env}})^s}(\boldsymbol{h}) = ((0, \alpha), i) \in \mathbb{N}^{2n,(s)}$*, then* $\boldsymbol{h} \notin \operatorname{Ker}(\mathfrak{m}^s)$ *and* $\exp_{R^s}(\mathfrak{m}^s(\boldsymbol{h})) = (\alpha^{\mathrm{op}}, i)$.

Using these results we have:

Theorem 7. *Let* $R = k\{x_1, \ldots, x_n; Q, \preceq\}$ *be a PBW algebra,* $M \subset R^s$ *be an R-bimodule and consider in* R^{env} *the PBW structure given in the Proposition 3 (where the order is one of* $\preceq^*, \preceq^c, \preceq_*$ *or* $\preceq_c$*).*

If G is a left Gröbner basis for $N_M = (\mathfrak{m}^s)^{-1}(M)$ *with TOP (resp. POT), then the set* $\mathfrak{m}^s(G) \setminus \{0\}$ *is a two-sided Gröbner basis for M with TOP (resp. POT).*

The theorem provides a method of construction of two-sided Gröbner bases for bimodules. For convenience we write this explicitly under the name of Algorithm 1.

Algorithm 1. Two-sided Gröbner bases

Require: $F = \{\boldsymbol{f_1}, \ldots, \boldsymbol{f_t}\} \subseteq R^s \setminus \{0\}$.
Ensure: $G = \{\boldsymbol{g_1}, \ldots, \boldsymbol{g_{t'}}\}$, a two-sided Gröbner basis for ${}_R\langle F\rangle_R$ such that $F \subseteq G$.

INITIALIZATION: $B := \{\boldsymbol{f_i} \otimes \mathbf{1}\}_{i=1}^t \cup \{\boldsymbol{x}^{(\epsilon_j,k)} \otimes \mathbf{1} - \mathbf{1} \otimes \boldsymbol{x}^{(\epsilon_j,k)}\}_{1\le j\le n,\ 1\le k\le s}$.
Using the left Buchberger algorithm, compute a left Gröbner basis G' in the free left R^{env}-module $(R^{\mathrm{env}})^s$ for the input data B.

If $G' = \{\boldsymbol{g'_1}, \ldots, \boldsymbol{g'_r}\}$ with $\boldsymbol{g'_i} = (\sum_{j\in\mathfrak{I}_i} p_{ij}^1 \otimes q_{ij}^1, \ldots, \sum_{j\in\mathfrak{I}_i} p_{ij}^s \otimes q_{ij}^s)$, take $\boldsymbol{g_i} := (\sum_{j\in\mathfrak{I}_i} p_{ij}^1 q_{ij}^1, \ldots, \sum_{j\in\mathfrak{I}_i} p_{ij}^s q_{ij}^s)$.

$G := \emptyset$.

for all $i = 1$ to r **do**
  if $\boldsymbol{g_i} \neq 0$ **then**

$G := G \cup \{g_i\}$.
end if
end for

The advantage offered by this algorithm is that only one call to the left Buchberger algorithm is made, whereas the one shown in Kandri-Rody and Weispfenning (1990) and Bueso et al. (2003) makes an, a priori, unknown number of calls. The left Buchberger algorithm can also be found in Kandri-Rody and Weispfenning (1990) and Bueso et al. (2003).

Example 8. Let R be the quantum plane, i.e., $R = \mathbb{C}\{x, y;\ \{yx - qxy\}, \preceq_{(1,3)}\}$, where $\preceq_{(1,3)}$ is the $(1,3)$-weighted lexicographical order with $\epsilon_1 \prec_{\text{lex}} \epsilon_2$, and put $q = i$. Let $F = \{(2x, x^2y, xy^2 + y^2), (xy, 0, -x^2y^2), (x^2, 2, 0)\} \subset R^3$ and consider the order TOP in R^3.

The old algorithm (implemented by the authors as it appears in Kandri-Rody and Weispfenning, 1990, page 12) calls the left Buchberger algorithm twice and takes 56.6 s to compute a (non-reduced) two-sided Gröbner basis G_1 consisting of 17 elements.

The Algorithm 1 takes 43.0 s to compute a (non-reduced) two-sided Gröbner basis G_2 with 12 elements.

After reducing G_1 or G_2, we obtain the reduced two-sided Gröbner basis

$$\{(x^2, 0, 0),\ (2x, 0, y^2),\ (xy, 0, 0),\ (0, 1, 0)\}$$

of ${}_R\langle F\rangle_R$. The reduction of G_1 takes 22.7 s whereas the reduction of G_2 takes 13.0 s.

Now consider the algebra $M_q(2) = \mathbb{C}\{x, y, z, t;\ Q, \preceq_{\text{glex}}\}$ of quantum matrices where $\preceq_{\text{glex}}$ denotes the degree lexicographical order with $\epsilon_1 \prec_{\text{glex}} \dots \prec_{\text{glex}} \epsilon_4$, and $Q = \{yx - qxy, ty - qyt, zx - qxz, tz - qzt, zy - yz, tx - xt - (q^{-1} - q)yz\}$. Put again $q = i$, and consider the order "POT" in R^2. Let $F = \{(-xzt + y, 2xy^3z), (x^2zt, y^2)\} \subset R^2$.

The old algorithm computes a two-sided Gröbner basis G_1 consisting of 28 elements in 167.6 s, calling the left Buchberger algorithm twice.

The Algorithm 1 takes 37.9 s to compute a two-sided Gröbner basis G_2 with 17 elements. The reduction of G_1 takes 101.5 s whilst the reduction of G_2 takes 9.8 s. The reduced two-sided Gröbner basis of ${}_R\langle F\rangle_R$ is

$$\{(xzt - y, 0),\ (xy, y^2),\ (0, y^3),\ (0, y^2z),\ (yz^2t, 0),\ (0, y^2t),\ (y^2, 0),\ (0, xy^2)\}.$$

The following table shows a comparison between both algorithms for these and some other explicit examples.

Module	Size of reduced	Old method			Algorithm 1		
		Size	Time	Red. time	Size	Time	Red. time
$\mathbb{C}_q[x, y]^3$	4	17	56.6	22.7	12	43.0	13.0
$M_q(2)^2$	8	28	167.6	101.5	17	37.9	9.8
$\mathfrak{D}$	6	38	1907.5	709.6	14	93.7	53.1
$U(\mathfrak{sl}(2))$	10	29	735.2	487.1	11	113.8	58.7
$U(\mathfrak{g}_2)$	14	88	24176.9	12583.0	46	13522.1	3728.7

The first column represents the free module where the computations are performed and the second represents the size of the reduced two-sided Gröbner basis for the corresponding example. For both algorithms, the column "Time" represents the time elapsed for computing a (not necessarily reduced) two-sided Gröbner basis, its number of elements being shown in the column "Size". We also give the time taken in addition to reduce the basis.

The first two rows gather the times and sizes of the examples described above.

The third row represents the computation of two-sided Gröbner bases in the *Diamond algebra* $\mathfrak{D} = \mathbb{C}\{x, y, z, t;\ \ Q, \preceq_{\text{lex}}\}$ where $\preceq_{\text{lex}}$ is the lexicographical order with $\epsilon_1 \prec_{\text{lex}} \ldots \prec_{\text{lex}} \epsilon_4$ and $Q = \{yx-xy, zx-xz, tx-xt, zy-yz+x, ty-yt+y, tz-zt-z\}$. The input data are $F = \{4x^2t + 5x^2y, 8z^3t + 9yz\}$. In this case, the old algorithm makes three calls to the left Buchberger algorithm in order to compute a two-sided Gröbner basis.

The fourth row is concerned with the example `AnnFD-sl2-2`, (see Levandovskyy and Schönemann, 2003) consisting in computing a two-sided Gröbner basis in the enveloping algebra of traceless 2×2-matrices $U(\mathfrak{sl}(2)) = k\{e, f, h;\ Q, \preceq_{\text{glex}}\}$, where $Q = \{fe-ef+h, he-eh-2e, hy-yh+2f\}$. The input data are $F = \{e^3, f^3, (h-2)h(h+2)\}$.

The last row of the table represents the results for the example `TwoGB-g2-2` described in Levandovskyy and Schönemann (2003). It consists in computing a two-sided Gröbner basis of the ideal generated by the square of the element x_1 of the algebra $U(\mathfrak{g}_2)$, which is generated by 14 elements.[1] Here we use the degree lexicographical order $\preceq_{\text{glex}}$ on $\mathbb{N}^{14}$.

4. Syzygy bimodules

In this section we study the notion of *syzygy bimodule* of a subset of R^s, where R is a PBW algebra and $s \in \mathbb{N}^*$. This notion can be viewed as the analogue of the *left syzygy module* for left modules, since it presents some similar properties.

There exists an algorithm (see again Bueso et al., 2003) which computes a generator system of the left syzygy module $Syz^l(F)$, provided a finite set of input data $F \subset R^s$ is given. This algorithm is shown below and will be used within Algorithm 3.

Algorithm 2. Left syzygy module

Require: $F = \{\boldsymbol{f_1}, \ldots, \boldsymbol{f_t}\} \subseteq R^s \setminus \{0\}$.
Ensure: H, a finite left generator system of $Syz^l(F)$.

INITIALIZATION: Run the left Buchberger algorithm for the input data F in order to compute:
- a left Gröbner basis $G = \{\boldsymbol{g_1}, \ldots, \boldsymbol{g_r}\} \subset R^s$ for ${}_R\langle F\rangle$,
- the elements $h_{ij}^k \in R$ such that $SP(\boldsymbol{g_i}, \boldsymbol{g_j}) = \sum_{k=1}^r h_{ij}^k \boldsymbol{g_k}$ for all $1 \le i < j \le r$, and
- the matrix $Q \in M_{r\times t}(R)$ such that $(\boldsymbol{g_1}, \ldots, \boldsymbol{g_r}) = (\boldsymbol{f_1}, \ldots, \boldsymbol{f_t})Q^t$.

for all $1 \le i < j \le r$ **do**

[1] See `http://www.singular.uni-kl.de/plural/DEMOS/Leipzig/Applications/G2/index.html` for a definition of $U(\mathfrak{g}_2)$ and a complete description of this example.

if level $(\boldsymbol{g_i})$ = level $(\boldsymbol{g_j})$ **then**

Compute r_{ij}, r_{ji} such that $SP(\boldsymbol{g_i}, \boldsymbol{g_j}) = r_{ij}\boldsymbol{g_i} - r_{ji}\boldsymbol{g_j}$.

Let $\boldsymbol{p_{ij}} := (0, \dots \overset{i}{r_{ij}} \dots, 0) - (0, \dots \overset{j}{r_{ji}} \dots, 0) - (h_{ij}^1, \dots, h_{ij}^r)$.

end if

end for

Let $H := \{\boldsymbol{p_{ij}}Q \,/\, 1 \le i < j \le r,$ and level (exp $(\boldsymbol{g_i})$) = level (exp $(\boldsymbol{g_j})$)$\}$.

Definition 9. Let $\boldsymbol{f_1}, \dots, \boldsymbol{f_t} \in R^s$. The *syzygy bimodule* of the matrix

$$F = \begin{bmatrix} \boldsymbol{f_1} \\ \vdots \\ \boldsymbol{f_t} \end{bmatrix} \in M_{t\times s}(R),$$

denoted by $Syz(F)$ or $Syz(\boldsymbol{f_1}, \dots, \boldsymbol{f_t})$, is the kernel of the homomorphism of left R^{env}-modules $(R^{\text{env}})^t \longrightarrow R^s;\ (h_1, \dots, h_t) \longmapsto \sum_{i=1}^t h_i \boldsymbol{f_i}$.

We can compute the syzygy bimodule of a matrix F using again the techniques shown in Section 3, that is, we can move the problem to the context of the enveloping algebra in order to use the methods on the left side.

Proposition 10. *Let $M \subseteq R^s$ be an R-bimodule and $N = (\mathfrak{m}^s)^{-1}(M)$. Let $\{\boldsymbol{h_1}, \dots, \boldsymbol{h_r}\} \subset (R^{\text{env}})^{t+sn}$ be a generator system of $Syz^l(\{\boldsymbol{f_i} \otimes \boldsymbol{1}\}_{i=1}^t, \{\boldsymbol{x^{(\epsilon_j,k)}} \otimes \boldsymbol{1} - \boldsymbol{1} \otimes \boldsymbol{x^{(\epsilon_j,k)}}\}_{1\le j\le n,\ 1\le k\le s})$ as a left R^{env}-module.*

Then $Syz(\boldsymbol{f_1}, \dots, \boldsymbol{f_t}) = {}_R\langle \pi(\boldsymbol{h_1}), \dots, \pi(\boldsymbol{h_r})\rangle_R$, where π is the projection homomorphism $\pi : (R^{\text{env}})^t \times (R^{\text{env}})^{sn} \longrightarrow (R^{\text{env}})^t$.

Proof. Note that $\sum_{i=1}^t g_i(\boldsymbol{f_i} \otimes \boldsymbol{1}) \in \text{Ker}(\mathfrak{m}^s)$ for any $\boldsymbol{g} = (g_1, \dots, g_t) \in Syz(\boldsymbol{f_1}, \dots, \boldsymbol{f_t})$. So, there exists an element $\boldsymbol{g'} = (g'_{11}, \dots, g'_{1s}, \dots, g'_{n1}, \dots, g_{ns}) \in (R^{\text{env}})^{sn}$ such that $\sum_i g_i(\boldsymbol{f_i} \otimes \boldsymbol{1}) = \sum_{j,k} g'_{jk}(\boldsymbol{x^{(\epsilon_j,k)}} \otimes \boldsymbol{1} - \boldsymbol{1} \otimes \boldsymbol{x^{(\epsilon_j,k)}})$. Hence, $(\boldsymbol{g}, -\boldsymbol{g'})$ is in $Syz^l(\{\boldsymbol{f_i} \otimes \boldsymbol{1}\}_{i=1}^t, \{\boldsymbol{x^{(\epsilon_j,k)}} \otimes \boldsymbol{1} - \boldsymbol{1} \otimes \boldsymbol{x^{(\epsilon_j,k)}}\}_{1\le j\le n,\ 1\le k\le s})$. From this point, the proof may easily be finished. □

Algorithm 3. Syzygy bimodule

Require: $F = \{\boldsymbol{f_1}, \dots, \boldsymbol{f_t}\} \subseteq R^s \setminus \{0\}$.
Ensure: H, a finite generator system of $Syz(F)$ as an R-bimodule.

INITIALIZATION: $B := \{\boldsymbol{f_i} \otimes \boldsymbol{1}\}_{i=1}^t \cup \{\boldsymbol{x^{(\epsilon_j,k)}} \otimes \boldsymbol{1} - \boldsymbol{1} \otimes \boldsymbol{x^{(\epsilon_j,k)}}\}_{1\le j\le n,\ 1\le k\le s}$.

Using the left syzygy module algorithm, compute in $(R^{\text{env}})^s$ a generator system $H = \{\boldsymbol{h_1}, \dots, \boldsymbol{h_r}\}$ of $Syz^l(B)$ as a left R^{env}-module.

If $\boldsymbol{h_i} = (\boldsymbol{h_i}', \boldsymbol{h_i}'')$ where $\boldsymbol{h_i}' \in (R^{\text{env}})^t$ and $\boldsymbol{h_i}'' \in (R^{\text{env}})^{sn}$ for $1 \le i \le r$, take $H := \{\boldsymbol{h_1}', \dots, \boldsymbol{h_r}'\}$.

Example 11. Let R be the quantum plane with the PBW algebra structure $\mathbb{C}\{x, y;\ \{yx - qxy\}, \preceq_{(2,1)}\}$, with $q = i$, and consider the order POT in R^2. Let $F = \{(x + 1, y), (xy, 0)\} \subset R^2$.

Algorithm 3 takes 13.4 s to compute the R-bimodule generator system H of $Syz(F)$ consisting of eight elements:

$$
\begin{aligned}
H = \Bigg\{ & (1 \otimes y - y \otimes 1, (-1+i)1 \otimes 1),\ \left(\left(-\frac{1}{2}+\frac{i}{2}\right) y \otimes x \right.\\
& \left. + \left(-\frac{1}{2}+\frac{i}{2}\right) xy \otimes 1, 1 \otimes x + 1 \otimes 1\right),\ (0, 1 \otimes y + iy \otimes 1),\\
& \left(\left(-\frac{1}{2}+\frac{i}{2}\right) y \otimes x + \left(-\frac{1}{2}+\frac{i}{2}\right) xy \otimes 1, ix \otimes 1 + 1 \otimes 1\right),\\
& \left(\left(-\frac{1}{2}-\frac{i}{2}\right) y \otimes x + \left(-\frac{1}{2}-\frac{i}{2}\right) xy \otimes 1, i \otimes x + i \otimes 1\right),\\
& (-y \otimes y + y^2 \otimes 1, i \otimes y - iy \otimes 1),\ \left(\left(\frac{1}{2}+\frac{i}{2}\right) 1 \otimes x^2 - ix \otimes x \right.\\
& \left. + \left(\left(-\frac{1}{2}+\frac{i}{2}\right) x^2 \otimes 1, 0\right)\right),\ \left(\frac{1}{2}+\frac{i}{2}\right) 1 \otimes xy + \left(-\frac{1}{2}+\frac{i}{2}\right) y \otimes x \\
& + \left(-\frac{1}{2}-\frac{i}{2}\right) x \otimes y + \left(\frac{1}{2}+\frac{i}{2}\right) xy \otimes 1, 0\Bigg)\Bigg\}.
\end{aligned}
$$

Although elimination techniques are useful in solving several problems in module theory (computation of intersections, quotient ideals, etc.), they appear to be computationally inefficient, mainly because elimination orders are unavoidably used.

On the other hand, it has been noted, first in the commutative case (cf. Adams and Loustaunau, 1994, page 171) and then using left syzygy R-modules where R is a non-commutative PBW algebra (cf. Bueso et al., 2003, page 203), that syzygies provide a more efficient treatment, for example, in the computation of the intersection of left R-submodules of R^s, ideal quotients, kernels of homomorphisms of left R-submodules, etc.

In what follows, we will see that some applications of left syzygies can be generalized using the new definition of syzygy bimodules, so that, for example, it is possible to give an algorithm to compute a finite intersection of R-subbimodules of R^s when, as is natural, two-sided input data are given. Further applications will be studied in following work.

The following result states a general property of the epimorphic image M of truncated two-sided syzygies.

Lemma 12. *Let M be an R-subbimodule of R^s such that there exist $p, q \geq 1$ and $H = [\frac{H_1}{H_2}] \in M_{(s+p)\times q}(R)$, where $H_1 \in M_{s\times q}(R)$ and $H_2 \in M_{p\times q}(R)$, satisfying the following two conditions:*

(i) $(\mathfrak{m}^s(\boldsymbol{h}) \otimes \mathbf{1})H_1 = \boldsymbol{h}H_1,\ \forall \boldsymbol{h} \in (R^{\text{env}})^s$.

(ii) $\{\boldsymbol{h} \in R^s;\ \exists \boldsymbol{h}'' \in (R^{\text{env}})^p$ such that $(\boldsymbol{h} \otimes \mathbf{1}, \boldsymbol{h}'') \in Syz(H)\} = M$.

Let us split up each element $\boldsymbol{h} \in (R^{\text{env}})^{s+p}$ into $\boldsymbol{h} = (\boldsymbol{h}', \boldsymbol{h}'')$ with $\boldsymbol{h}' \in (R^{\text{env}})^s$ and $\boldsymbol{h}'' \in (R^{\text{env}})^p$.

(1) *If* $\{\boldsymbol{h_1}, \ldots, \boldsymbol{h_t}\} \subseteq (R^{\mathrm{env}})^{s+p}$ *is an R-bimodule generator system of* $Syz(H)$*, then* $M = {}_R\langle \mathfrak{m}^s(\boldsymbol{h_1}'), \ldots, \mathfrak{m}^s(\boldsymbol{h_t}')\rangle_R$.

(2) *Furthermore, if* $\{\boldsymbol{h_1}, \ldots, \boldsymbol{h_t}\}$ *is a left Gröbner basis of* $Syz(H)$ *(as a left* R^{env}*-module) for the order POT in* $(R^{\mathrm{env}})^{s+p}$ *and any of* $\preceq^*, \preceq^c, \preceq_*, \preceq_c$ *in* R^{env}*, then* $\{\mathfrak{m}^s(\boldsymbol{h_1}'), \ldots, \mathfrak{m}^s(\boldsymbol{h_t}')\} \setminus \{0\}$ *is a two-sided Gröbner basis of M for POT.*

Proof. First, assume $\{\boldsymbol{h_1}, \ldots, \boldsymbol{h_t}\}$ is a generator system of $Syz(H)$ as an R-bimodule, where $\boldsymbol{h_i} = (\boldsymbol{h_i}', \boldsymbol{h_i}'') \in (R^{\mathrm{env}})^s \times (R^{\mathrm{env}})^p$. Then,

$$(\mathfrak{m}^s(\boldsymbol{h_i}') \otimes \mathbf{1})H_1 + \boldsymbol{h_i}''H_2 = \boldsymbol{h_i}'H_1 + \boldsymbol{h_i}''H_2 = 0,$$

for all $1 \le i \le t$. Therefore, ${}_R\langle \mathfrak{m}^s(\boldsymbol{h_1}'), \ldots, \mathfrak{m}^s(\boldsymbol{h_t}')\rangle_R \subseteq M$. Conversely, for all $\boldsymbol{f} \in M$, there exists $\boldsymbol{h}'' \in (R^{\mathrm{env}})^p$ such that $(\boldsymbol{f} \otimes \mathbf{1}, \boldsymbol{h}'') \in Syz(H)$. Hence

$$(\boldsymbol{f} \otimes \mathbf{1}, \boldsymbol{h}'') = \sum_{i=1}^{t} p_i(\boldsymbol{h_i}', \boldsymbol{h_i}'')$$

for some p_i in R^{env}, $1 \le i \le t$, and applying $\mathfrak{m}^s$ to the first s components,

$$\boldsymbol{f} = \mathfrak{m}^s(\boldsymbol{f} \otimes \mathbf{1}) = \mathfrak{m}^s\left(\sum_{i=1}^{t} p_i\boldsymbol{h_i}'\right) = \sum_{i=1}^{t} p_i\mathfrak{m}^s(\boldsymbol{h_i}').$$

In order to prove the second statement, first suppose that the order on R^{env} is either $\preceq^*$ or $\preceq^c$. Pick $\boldsymbol{f} \in M \setminus \{0\}$. Then there exists $\boldsymbol{h}'' \in (R^{\mathrm{env}})^p$ such that $(\boldsymbol{f} \otimes \mathbf{1}, \boldsymbol{h}'') \in Syz(H)$. Moreover, as $\boldsymbol{f} \otimes \mathbf{1} \neq 0$ and the order on $(R^{\mathrm{env}})^{s+p}$ is POT,

$$\exp_{(R^{\mathrm{env}})^s}(\boldsymbol{f} \otimes \mathbf{1}) = \exp_{(R^{\mathrm{env}})^{s+p}}((\boldsymbol{f} \otimes \mathbf{1}, \boldsymbol{h}'')) = (\alpha_1, \alpha_2) + \exp_{(R^{\mathrm{env}})^{s+p}}(\boldsymbol{h_j}),$$

for some $j \in \{1, \ldots, t\}$ and $\alpha_1, \alpha_2 \in \mathbb{N}^n$. The level of that element $\boldsymbol{h_j}$ is the level of $\boldsymbol{f} \otimes \mathbf{1}$, which is in $\{1, \ldots, s\}$, so $\boldsymbol{h_j}' \neq 0$ and $\exp_{(R^{\mathrm{env}})^{s+p}}(\boldsymbol{h_j}) = \exp_{(R^{\mathrm{env}})^s}(\boldsymbol{h_j}')$. Therefore,

$$\exp_{(R^{\mathrm{env}})^s}(\boldsymbol{f} \otimes \mathbf{1}) = (\alpha_1, \alpha_2) + \exp_{(R^{\mathrm{env}})^s}(\boldsymbol{h_j}'). \tag{1}$$

Now, if $\exp_{R^s}(\boldsymbol{f}) = (\beta, k) \in \mathbb{N}^{n,(s)}$, then $\exp_{(R^{\mathrm{env}})^s}(\boldsymbol{f} \otimes \mathbf{1}) = ((\beta, 0), k) \in \mathbb{N}^{2n,(s)}$. So, if $((\gamma_1, \gamma_2), k)$ is the exponent of $\boldsymbol{h_j}'$, then the Eq. (1) can be rewritten as

$$((\beta, 0), k) = (\alpha_1, \alpha_2) + ((\gamma_1, \gamma_2), k), \tag{2}$$

which leads us to $\alpha_2 = \gamma_2 = 0$, and hence, $\exp_{(R^{\mathrm{env}})^s}(\boldsymbol{h_j}') = ((\gamma_1, 0), k)$. By Lemma 6, this implies $\exp_{R^s}(\mathfrak{m}^s(\boldsymbol{h_j}')) = (\gamma_1, k)$. Finally, extracting the first component of the pairs in Eq. (2),

$$\exp_{R^s}(\boldsymbol{f}) = (\beta, k) = (\alpha_1 + \gamma_1, k) = \alpha_1 + \exp_{R^s}(\mathfrak{m}^s(\boldsymbol{h_j}')).$$

If the order on R^{env} is $\preceq_*$ or $\preceq_c$, the proof follows analogously. $\square$

Theorem 13. *Let* $\{M_i\}_{i=1}^r$ *be a family of* R*-subbimodules of* R^s *and suppose that* $M_i = {}_R\langle \boldsymbol{f_1^i}, \ldots, \boldsymbol{f_{t_i}^i}\rangle_R \subseteq R^s$. *Then*

$$\bigcap_{i=1}^r M_i = \{\boldsymbol{h} \in R^s;\ \exists \boldsymbol{h}'' \in (R^{\mathrm{env}})^{\sum_{j=1}^r t_j} \text{ such that } (\boldsymbol{h} \otimes \mathbf{1}, \boldsymbol{h}'') \in Syz(H)\},$$

where

$$H = \begin{pmatrix} I_s & \cdots & I_s \\ \boldsymbol{f_1^1} & \cdots & \mathbf{0} \\ \vdots & & \vdots \\ \boldsymbol{f_{t_1}^1} & \cdots & \mathbf{0} \\ \vdots & & \vdots \\ \mathbf{0} & \cdots & \boldsymbol{f_1^r} \\ \vdots & & \vdots \\ \mathbf{0} & \cdots & \boldsymbol{f_{t_r}^r} \end{pmatrix} \in M_{(s+\sum_{i=1}^r t_i)\times rs}(R)$$

Proof. If $(\boldsymbol{h} \otimes \mathbf{1}, \boldsymbol{h}'') \in Syz(H)$, then, for $1 \le i \le r$,

$$\boldsymbol{h} = \sum_{l=1}^s h_l \otimes 1 \cdot \boldsymbol{e_l} = -\sum_{k=1}^{t_i} h''_{k+\sum_{j<i} t_j} \boldsymbol{f_k^i} \in M_i.$$

Conversely, if $\boldsymbol{h} \in \bigcap_{i=1}^r M_i$, then $\boldsymbol{h} = \sum_{k=1}^{t_i} h_k^i \boldsymbol{f_k^i}$ with $h_k^i \in R^{\mathrm{env}}$, for all $i \in \{1, \ldots, r\}$. Therefore,

$$(\boldsymbol{h} \otimes \mathbf{1}, -h_1^1, \ldots, -h_{t_1}^1, \ldots, -h_1^r, \ldots, -h_{t_r}^r) \in Syz(H). \quad \square$$

From 12 and 13, an algorithm for computing finite intersections of R-subbimodules of R^s may be formulated (see Algorithm 4).

Corollary 14. *Let* M_i *and* H *be as in Theorem 13. If* $Syz(H) = {}_R\langle \boldsymbol{g_1}, \ldots, \boldsymbol{g_t}\rangle_R$ *with* $\boldsymbol{g_k} = (\boldsymbol{g_k}', \boldsymbol{g_k}'') \in (R^{\mathrm{env}})^s \times (R^{\mathrm{env}})^{\sum_{j=1}^r t_j}$ *for all* $1 \le k \le t$, *then*

$$\bigcap_{i=1}^r M_i = {}_R\langle \mathfrak{m}^s(\boldsymbol{g_1}'), \ldots, \mathfrak{m}^s(\boldsymbol{g_t}')\rangle_R.$$

If $G = \{\boldsymbol{g_1}, \ldots, \boldsymbol{g_t}\}$ *is in addition a left Gröbner basis of* $Syz(H)$ *(as left* R^{env}*-module) with POT in* $(R^{\mathrm{env}})^{s+\sum_{j=1}^r t_j}$, *then* $\{\mathfrak{m}^s(\boldsymbol{g_1}'), \ldots, \mathfrak{m}^s(\boldsymbol{g_t}')\} \setminus \{0\}$ *is a two-sided Gröbner basis of* $\bigcap_{i=1}^r M_i$ *with POT in* R^s.

Algorithm 4. Intersection of R-subbimodules of R^s

Require: $\{M_i\}_{i=1}^r$, a family of R-subbimodules of R^s with $M_i = {}_R\langle \boldsymbol{f_1^i}, \ldots, \boldsymbol{f_{t_i}^i}\rangle_R$.
Ensure: M, a finite generator system of $\bigcap_{i=1}^r M_i$ as an R-bimodule.

INITIALIZATION:

$$H := \begin{pmatrix} I_s & \cdots & I_s \\ \boldsymbol{f_1^1} & \cdots & \mathbf{0} \\ \vdots & & \vdots \\ \boldsymbol{f_{t_1}^1} & \cdots & \mathbf{0} \\ \vdots & & \vdots \\ \mathbf{0} & \cdots & \boldsymbol{f_1^r} \\ \vdots & & \vdots \\ \mathbf{0} & \cdots & \boldsymbol{f_{t_r}^r} \end{pmatrix} \in M_{(s+\sum_{j=1}^r t_j)\times rs}(R).$$

Using the syzygy bimodule algorithm compute a generator system $G = \{\boldsymbol{g_1}, \ldots, \boldsymbol{g_t}\}$ of $Syz(H)$ as an R-bimodule.

If $\boldsymbol{g_k} = (\boldsymbol{g_k}', \boldsymbol{g_k}'')$ where $\boldsymbol{g_k}' \in (R^{\text{env}})^s$ and $\boldsymbol{g_k}'' \in (R^{\text{env}})^{\sum_{j=1}^r t_j}$ for $1 \le k \le t$,

take $M := \{\mathfrak{m}^s(\boldsymbol{g_1}'), \ldots, \mathfrak{m}^s(\boldsymbol{g_t}')\}$.

Example 15. Let R be the quantum plane (as in Example 11) and consider the order POT in R^2. Let M_1 and M_2 be the R-subbimodules of R^2 generated by $\{(2x^2 + 2x, -y), (0, -8), (-3xy, 0)\}$ and $\{(x + 2, 0), (1, -y)\}$, respectively.

Algorithm 4 takes 109.1 s to compute the R-bimodule generator system M of $M_1 \cap M_2$, consisting of 12 elements:

$$\begin{aligned} M = \Bigg\{ & \left(\frac{4i}{3}x^2y + \frac{7}{3}xy, \frac{2i}{3}y^2\right), \left(-\frac{2i}{3}x^3 + \left(-\frac{4}{3} - \frac{2i}{3}\right)x^2 - \frac{4}{3}x, \frac{i}{3}xy + \frac{2}{3}y\right), \\ & \left(-\frac{5}{3}x^2y + \frac{8i}{3}xy, -\frac{4}{3}y^2\right), \left(\left(-1 - \frac{5i}{3}\right)x^3 + \left(-\frac{19}{3} - \frac{5i}{3}\right)x^2 \right. \\ & \left. - \frac{16}{3}x, \frac{4i}{3}xy + \frac{8}{3}y\right), \left(\frac{10}{3}x^2y + \left(-1 - \frac{13i}{3}\right)xy, \frac{8}{3}y^2\right), \\ & \left(\frac{2}{3}x^2 + \frac{2}{3}x, -\frac{4}{3}y\right), \left(\frac{4}{3}x^2y^2 - \frac{7}{3}xy^2, -\frac{2}{3}y^3\right), \left(-\frac{2}{3}x^3 - 2x^2 \right. \\ & \left. - \frac{4}{3}x, \frac{1}{3}xy + \frac{2}{3}y\right), \left(\left(\frac{4}{3} + \frac{4i}{3}\right)x^2y + \left(\frac{4}{3} - \frac{4i}{3}\right)xy, \left(\frac{2}{3} + \frac{2i}{3}\right)y^2\right), \\ & \left(\frac{-2i}{3}x^3y - 2x^2y + \frac{4i}{3}xy, -\frac{i}{3}xy^2 - \frac{2}{3}y^2\right), \left(\left(-\frac{2}{3} + \frac{2i}{3}\right)x^3 \right. \\ & \left. + \left(-\frac{2}{3} + \frac{2i}{3}\right)x^2, 0\right), \left(-\frac{4}{3}x^2y + \left(-\frac{2}{3} + \frac{2i}{3}\right)xy, 0\right) \Bigg\}. \end{aligned}$$

Acknowledgements

The authors are grateful to F. Castro for motivation, and to P. Bueso and A. Verschoren for their corrections and suggestions.

References

Adams, W.W., Loustaunau, P., 1994. An Introduction to Gröbner Bases. In: AMS Grad. Studies in Math., vol. 3.

Apel, J., Lassner, W., 1985. An algorithm for calculations in enveloping fields of Lie algebras. In: Proc. Conf. Comp. Alg. Dubna. pp. 231–241.

Bergman, G., 1978. The diamond lemma in ring theory. Adv. Math. 29, 178–218.

Bueso, J.L., Castro, F., Gómez Torrecillas, J., Lobillo, F.J., 1996. Computing the Gelfand–Kirillov dimension. SAC Newsletter 1, 39–52.

Bueso, J.L., Castro, F., Gómez Torrecillas, J., Lobillo, F.J., 1998. An introduction to effective calculus in quantum groups. Lect. Notes Pure Appl. Math. 197, 55–83.

Bueso, J.L., Castro, F., Gómez Torrecillas, J., Lobillo, F.J., 1999. Test de primalité dans une extension de Ore itérée. C. R. Acad. Sci. Paris Sér. I Math. 328, 459–462.

Bueso, J.L., Gómez Torrecillas, J., Verschoren, A., 2003. Algorithmic Methods in Non-commutative Algebra: Applications to Quantum Groups. Kluwer.

Gago-Vargas, J., 2003. Bases for projective modules in $A_n(k)$. J. Symbolic Comput. 36, 845–853.

Galligo, A., 1982. Algorithmes de cacul de bases standard, preprint Niza.

García Román, M.D., García Román, M.S., 2001. Un nuevo algoritmo para el cálculo de bases de Gröbner biláteras. In: Proc. EACA'2001 held in La Rioja-Spain (in Spanish). There is an accessible abstract on http://csimbolico.rediris.es/eaca01/20garcia.pdf.

Kandri-Rody, A., Weispfenning, V., 1990. Non-commutative Gröbner bases in algebras of solvable type. J. Symbolic Comput. 9, 1–26.

Kobayashi, Y., 2004. Gröbner bases of associative algebras and the Hochschild cohomology. Trans. Amer. Math. Soc. (electronic, posted July 16).

Kredel, H., 1993. Solvable Polynomial Rings. Shaker.

Levandovskyy, V., 2003. PBW bases, non-degeneracy conditions and applications. In: Buchweitz, R.-O., Lenzing, H (Eds.), Proceedings of the ICRA X Conference, Toronto, Canada. Fields Inst. Commun. 43.

Levandovskyy, V., Schönemann, H., 2003. Plural—a computer algebra system for noncommutative polynomial algebras. In: Proceedings of ISSAC 2003. ACM.

Li, H., 2002. Noncommutative Gröbner Bases and Filtered–Graded Transfer. In: Springer Lect. Notes in Math., vol. 1795.

Lobillo, F.J., 1998. Métodos algebraicos y efectivos en grupos cuánticos. Ph.D. Thesis, Universidad de Granada (in Spanish).

Pesch, M., 1998. Two-sided Gröbner bases in iterated Ore extensions. In: Symbolic Rewriting Techniques. Papers from the Workshop held in Ascona, Switzerland, April 30–May 4, 1995. Prog. Comput. Sci. Appl. Log. 15, 225–243.