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Nonclassical reductions of a 3+1-cubic nonlinear Schrödinger system

Elizabeth L. Mansfield [a,1], Gregory J. Reid [b,2], Peter A. Clarkson [a,3]

[a] *Institute of Mathematics and Statistics, University of Kent, Canterbury, CT2 7NF, UK*

[b] *Centre for Experimental and Computational Mathematics, Simon Fraser University, Vancouver, Canada V5A 1S6*

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Abstract

An analytical study, strongly aided by computer algebra packages `diffgrob2` by Mansfield and `rif` by Reid, is made of the 3+1-coupled nonlinear Schrödinger (CNLS) system $i\Psi_t+\nabla^2\Psi+\left(|\Psi|^2+|\Phi|^2\right)\Psi=0$, $i\Phi_t+\nabla^2\Phi+\left(|\Psi|^2+|\Phi|^2\right)\Phi=0$. This system describes transverse effects in nonlinear optical systems. It also arises in the study of the transmission of coupled wave packets and "optical solitons", in nonlinear optical fibres.

First we apply Lie's method for calculating the classical Lie algebra of vector fields generating symmetries that leave invariant the set of solutions of the CNLS system. The large linear classical determining system of PDE for the Lie algebra is automatically generated and reduced to a standard form by the `rif` algorithm, then solved, yielding a 15-dimensional classical Lie invariance algebra.

A generalization of Lie's classical method, called the nonclassical method of Bluman and Cole, is applied to the CNLS system. This method involves identifying nonclassical vector fields which leave invariant the joint solution set of the CNLS system and a certain additional system, called the invariant surface condition. In the generic case the system of determining equations has 856 PDE, is nonlinear and considerably more complicated than the linear classical system of determining equations whose solutions it possesses as a subset. Very few calculations of this magnitude have been attempted due to the necessity to treat cases, expression explosion and until recent times the dearth of mathematically rigorous algorithms for nonlinear systems.

The application of packages `diffgrob2` and `rif` leads to the explicit solution of the nonclassical determining system in eleven cases. Action of the classical group on the nonclassical vector fields considerably simplifies one of these cases. We identify the reduced form of the CNLS system in each case. Many of the cases yield new results which apply equally to a generalized coupled nonlinear Schrödinger system in which $|\Psi|^2+|\Phi|^2$ may be replaced by an arbitrary function of $|\Psi|^2+|\Phi|^2$. Coupling matrices in $\mathfrak{sl}(2,\mathbb{C})$ feature prominently in this family of reductions. © 1998 Elsevier Science B.V.

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[1] E-mail: E.L.Mansfield@ukc.ac.uk

[2] E-mail: reid@cecm.sfu.ca

[3] E-mail: P.A.Clarkson@ukc.ac.uk

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1. Introduction

In this paper we are concerned with the 3+1-dimensional coupled nonlinear Schrödinger (CNLS) system

$$\begin{aligned} &\mathrm{i}\Psi_t + \nabla^2\Psi + \left(|\Psi|^2 + |\Phi|^2\right)\Psi = 0\,, \\ &\mathrm{i}\Phi_t + \nabla^2\Phi + \left(|\Psi|^2 + |\Phi|^2\right)\Phi = 0\,, \end{aligned} \tag{1.1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and Ψ and Φ are complex-valued functions of x, y, z and t. Below we discuss various subcases of this equation which have been of interest in the areas of propagation of pulses in optical fibres and transverse effects in nonlinear optical systems [3,56,57,62,63,76,81]. They have been of particular interest also in the field of "optical solitons" [1,4,37,38].

In recent years there have been many studies of the propagation of pulses in nonlinear optical fibres. In a single mode optical fibre, the nonlinear Schrödinger equation

$$\mathrm{i}\psi_t + \tfrac{1}{2}\psi_{xx} + |\psi|^2\psi = 0 \tag{1.2}$$

governs the pulse propagation in a polarization-preserving, nonlinear optical fibre with cubic "Kerr-type" nonlinearity. Typically in optical applications t and x are replaced with z and τ. The nonlinear Schrödinger equation (1.2) is a well-known completely integrable equation solvable by inverse scattering [86] which possesses soliton solutions. Since solitons can propagate without dispersion, it has been proposed that they be used in long-distance optical communication systems.

For a nonlinear optical fibre with birefringence, the evolution of two polarization envelopes is governed by a system of coupled nonlinear Schrödinger equations which in nondimensional form are

$$\begin{aligned} &\mathrm{i}\psi_\xi + \mathrm{i}\delta\psi_\tau - \kappa\psi + \tfrac{1}{2}\psi_{xx} + \tfrac{1}{2}\psi_{yy} + \tfrac{1}{2}\psi_{\tau\tau} + \left(|\psi|^2 + \alpha|\phi|^2\right)\psi + \beta\phi^2\psi^* = 0\,, \\ &\mathrm{i}\phi_\xi - \mathrm{i}\delta\phi_\tau + \kappa\phi + \tfrac{1}{2}\phi_{xx} + \tfrac{1}{2}\phi_{yy} + \tfrac{1}{2}\phi_{\tau\tau} + \left(\alpha|\psi|^2 + |\phi|^2\right)\phi + \beta\psi^2\phi^* = 0 \end{aligned}$$

([56], see also [80,81]), where * denotes complex conjugation. Here ψ and ϕ denote the pulse envelopes along the orthogonal fast and slow birefringence axes, τ is the normalised time, and ξ the normalised spatial distance along the fibre axis. The coefficients α and β represent the strength of the nonlinear coupling, which by symmetry arguments satisfy $\alpha+\beta = 1$. The parameters κ, δ and β are the phase velocity, group velocity and nonlinear birefringence coefficients, respectively (for further details, see, for example, [4] and the references therein).

There have been a number of studies of the 1+1-dimensional coupled nonlinear Schrödinger equations

$$\begin{aligned} &\mathrm{i}\psi_t + \psi_{xx} + \sigma\left(|\psi|^2 + \alpha|\phi|^2\right)\psi = 0\,, \\ &\mathrm{i}\phi_t + \phi_{xx} + \sigma\left(\alpha|\psi|^2 + |\phi|^2\right)\phi = 0\,, \end{aligned} \tag{1.3}$$

where $\sigma = \pm 1$ and α is a constant. This equation can be derived from Maxwell's equations (cf. [49,57,62,76]) and exact solutions of (1.3) are discussed in [57,62,63]. The integrability of the system (1.3) has been discussed by Radhakrishnan, Sahadevan and Lakshmanan [67] (see also [6,77]) who show that the system (1.3) satisfies the Painlevé test due to Weiss, Tabor & Carnevale [83] only if $\alpha = 1$. In this case (1.3) was shown to be solvable by inverse scattering by Manakov [48] and Zakharov and Schulman [85]. Some recent studies of soliton solutions of (1.3) with $\alpha = 1$ include [45,65,66].

Typically solitary wave and one soliton solutions of PDE arise from classical symmetry reductions as travelling wave solutions, while N soliton solutions arise from classical reductions corresponding to generalized (Lie Bäcklund) symmetries. We discuss symmetry reductions in the next section and nonclassical reductions in the one that follows. In this paper we find a rich supply of nonclassical reductions. Determination of properties of exact solutions arising from these reductions, and whether any correspond to solitary wave solutions or solitons not obtainable by the classical methods, is an important problem for future work.

1.1. Symmetry reductions

Increasingly, fully nonlinear qualitative, exact analytic and asymptotic methods are used to explore the range of phenomena possessed by solutions of nonlinear equations. Unfortunately, mathematical techniques applicable to wide classes of nonlinear differential equations are few. One such method, the use of classical Lie symmetries to find canonical forms of equations, coordinates in which scaling similarities exist, and reductions and special exact solutions, is increasingly popular. Computer algebra relieves the researcher from the excessive calculations involved in obtaining Lie symmetries of an equation in the initial stages of its investigation. Exact solutions in terms of familiar special functions are more simply analysed for qualitative behaviours than large numerical data sets. Moreover, the effects of varying parameters can be more cheaply investigated.

Much effort has been expended on the use of adaptive grid methods for differential equations which are invariant under the action of a symmetry group. Numerical methods which do not reflect the symmetry structures of the partial differential equations they are approximating can exhibit poor performance in the long-term behaviour of the methods or in the computation of singularities. The reason is that solutions invariant under a symmetry often act as global attractors for more general solutions, with a variety of initial data, and accurately reflect the intermediate asymptotic behaviour after the initial effects have died down and before boundary effects become important [5]. Numerical methods which compute the natural variables governing the evolution of the solution, such as adaptive time-stepping on a moving mesh, can be more effective than the standard methods.

Exact solutions are useful not only in the design of numerical code but also in their testing. Exact solutions for nonlinear equations are rare, and methods which can generate families of them are not only increasingly popular, but increasingly sought.

If the Lie algebra of symmetries leaving a PDE (or system) invariant has been identified, then it is possible to seek the subset of its solutions invariant under a member of its Lie algebra. The classical vector field vanishes on these solutions, and this yields an auxiliary first order PDE or system called the Invariant Surface Condition (ISC). Integration of the ISC and substitution into the original PDE yields its reduced form under the given vector field. Such symmetry reductions, which are generally more analytically tractable than the original differential equation, have been used extensively in applications.

Gagnon [29] studied symmetry reductions of the 2+1-dimensional coupled nonlinear Schrödinger equations,

$$\begin{aligned} &\mathrm{i}\psi_t + \psi_{xx} + \psi_{yy} + \sigma\left(|\psi|^2 + \alpha|\phi|^2\right)\psi = 0\,, \\ &\mathrm{i}\phi_t + \phi_{xx} + \phi_{yy} + \sigma\left(\alpha|\psi|^2 + |\phi|^2\right)\phi = 0\,, \end{aligned}$$

where $\sigma = \pm 1$ and α is a constant. Gagnon also obtained some exact and approximate solutions. Parker [62] has a discussion of symmetry reductions of the associated 1+1-dimensional system. The model is of particular interest in the field of transverse effects in nonlinear optical systems (cf. [3] and the references therein). Recently Sciarrino and Winternitz [78] studied Lie symmetries of the vector nonlinear Schrödinger equation

$$\mathrm{i}\boldsymbol{\phi}_t + \boldsymbol{\phi}_{xx} + \boldsymbol{\phi}_{yy} + |\boldsymbol{\phi}|^2\boldsymbol{\phi} = 0\,,$$

where $\boldsymbol{\phi}$ is either a two or three component complex vector.

In a series of papers, Gagnon and Winternitz [30,32–36] (see also [28,31]) studied symmetry reductions of the 3+1-dimensional nonlinear Schrödinger equation

$$\mathrm{i}\psi_t + \psi_{xx} + \psi_{yy} + \psi_{zz} + |\psi|^2\psi = 0\,. \tag{1.4}$$

This system arises in the Landau–Ginzburg model of phase transitions [24] and in nonlinear optics. It also arises in the propagation of slowly varying electromagnetic wave envelopes in a plasma.

1.2. Nonclassical reductions

It has been recognised for some time that similarity solutions obtainable by classical Lie symmetries do not give the full range of possible invariant solutions. There are now many studies in the literature that seek to generalise classical Lie symmetry analysis, and the finding of new exact solutions to nonlinear equations in particular. A sampler of such methods with physically significant examples has been given by Olver and Vorob'ev [44], Chapter 11.

In this article we are particularly interested in the so-called *Nonclassical Method* of reduction due to Bluman and Cole [9]. Interest in the method rekindled in 1989 when it was realised that the catalogue of new exact solutions to the Boussinesq equation found by the *Direct Method* of Clarkson and Kruskal [20] could also be obtained by the Nonclassical Method [46]. It has now been shown that the Nonclassical Method is more general than the Direct Method [60]. Many authors have successfully used the method to obtain new, exact and interesting solutions to equations of mathematical and physical interest, and we refer the reader to the reviews [19,21,27].

The nonclassical method, in contrast to Lie's classical method, seeks so-called nonclassical vector fields which leave invariant the *joint* solution set of the original PDE *and* its invariant surface condition. Naturally any classical vector field is also a nonclassical vector field, since classical vector fields leave the ISC invariant. However, there are many examples of genuine nonclassical vector fields which do not leave the entire solution set of the original PDE invariant. Just as in the classical method, one can obtain reduced solutions of the original PDE. For genuine nonclassical vector fields these generally cannot be obtained by Lie's classical method. Further discussion of the method is given in Section 3.

One of our motivations for seeking nonclassical reductions of (1.1) is that Clarkson [18] had demonstrated that the 3+1-dimensional nonlinear Schrödinger equation (1.4) possesses symmetry reductions that are not obtainable using the classical Lie method.

1.3. Computer algebra packages

The calculations in this paper made considerable use of computer algebra packages written by the authors which analyse overdetermined systems of nonlinear differential equations. Indeed, the results given here would not have been possible without the use of computer algebra. For the two main nonclassical cases, systems of 856 and 238 nonlinear differential equations for the infinitesimals needed to be solved. The packages used, `rif` by Wittkopf and `diffgrob2` by Mansfield, seek integrability and compatibility conditions of overdetermined systems in a systematic manner. The output is a system equivalent to the input, in that the analytic solutions are the same, but which is either in a *reduced involutive form* or a *differential Gröbner basis* respectively, subject to the restrictions of the available theory.

Algorithms to complete systems to involutive form were developed by Janet, Riquier, Vessiot, Cartan and others, beginning a century ago (a modern explanation can be seen in [79]). The so-called Formal Theory of Differential Systems of Spencer and co-authors grew out of such considerations. Modern work following on from the ideas of Cartan can be seen in [12]. Recent authors have adapted advances in the analysis of nonlinear algebraic systems, especially the theory of Gröbner bases [7] and the algorithm to obtain a Gröbner basis due to Buchberger [13], to differential systems; see for example [11,15,47,54,61]. There have been a variety of adaptations, as there is more than one possible way of viewing a differential system as an algebraic system. Some ideas are more practical than others, some suitable only for particular applications or for particular types of systems, some of purely theoretical interest. There are many open problems. However, the best future results in the development of algorithms will come from testing the limits of those that we have, in applications such as the present one.

The utility of casting a system into an involutive or similar equivalent form is well known. The initial application, due to Riquier, was for obtaining formal series solutions which could be obtained coherently and which converged. Applications of Riquier's theorem to symmetries of differential equations have been given in [68–70]. The classical algorithms have been modernised, improved and implemented in the MAPLE package `standard_form` [72].

Reid et al. [73] also give an algorithmic approach to nonlinear systems which incorporates aspects from both geometric and algebraic approaches (see [74] for a theoretical exposition). This approach has been implemented by Wittkopf in his MAPLE package `rif`.

The calculations for a differential Gröbner basis can be seen to occur in these classical algorithms, albeit cast in a different form, and the theoretical advantages of a Gröbner basis for various applications can often be carried over to differential systems. For example, by adapting lexicographic term orderings in algebraic Gröbner basis theory, systematic eliminations can be achieved. Thus, one can answer questions such as "what are the equations, implied by the input system, for a given subset of the dependent variables?", or, "are there any equations, implied by the input system, in which derivatives with respect to just one of the independent variables appear?" Such questions can be answered definitively for linear systems. The results for nonlinear systems, obtainable by the algorithms implemented in the MAPLE package `diffgrob2` used here, are given in [22] and discussed in [50,54]. With an elimination term ordering, the output of the algorithm is called a triangulation of the system, as it is reminiscent of the triangulation or echelon form of a matrix. And the output can be solved in much the same way; from the bottom up. For examples of what can be achieved using our differential Gröbner bases, we refer the reader to [50–53]. In addition, for systems of PDE with a finite number of parameters in their general solutions, elimination orderings can reduce the integration of the system to the integration of ODE. Further, algorithms can be given for finding all rational solutions when the input systems are linear and have rational function coefficients [71].

The packages used in this paper followed two different philosophies. The `rif` algorithm and its predecessor `standard_form` are based on an automatic philosophy. While `diffgrob2` can be used automatically, especially on linear systems, complex nonlinear systems are better analysed interactively. The difficulty of analysing large or complex systems should not be underestimated, and both approaches have their advantages and disadvantages. We stress that although both packages implement algorithms, the algorithms allow for many choices at each stage. The theorems governing the properties of the output are independent of these choices, but efficiency, expression swell and memory use are not independent. Thus, the development of heuristics to guide the choices is an important open problem.

Using `diffgrob2` on a highly nonlinear system, typically one seeks at first, interactively, simple integrability conditions. These usually lead to substantial simplification of the analysis of the remaining equations. Semi-algorithmic methods for seeking such conditions which have worked in practice are discussed in the manual. Automatic procedures are best invoked on simple subsystems and the results used to simplify the remaining equations. The choice of term ordering may be crucial in controlling memory swell; experimenting with the term ordering is easily done in `diffgrob2`. If an integrability condition is obtained which factors, then each factor leads to a separate branch of the solution set. The automatic procedures in `diffgrob2` have an option which allows one to choose the desired factor of every condition obtained. A simple exposition of the theoretical principles underpinning the algorithms, which must be followed to guide interactive use of the package, can be found in [53].

Judicious interactive integration of some, but not all, simple subsystems en route may yield substantial simplification. We warn that the introduction of functions of integration generally increases the number of dependent variables. It can also change the type of functions in the system, from functions easily handled by computer algebra, such as polynomial functions, to ones which are not, for example, combinations of trigonometric and root functions.

The `rif` and `standard_form` algorithms employ a primarily automatic approach. Integration is generally not used and the package has a number of automatic strategies for picking out simple subsystems. For example the greedy strategy attempts to reduce the entire system at once, and can often lead to memory explosion. The conservative option reduces the current subsystem, together with what it views as the simplest equation of the rest of the system, and continues until the whole system is in `rif` or `standard` form. One disadvantage of such an automatic philosophy is that its criteria for a simple equation may be too rigid. For example, conditions may be missed which would be easily recognized by an experienced human as a key to substantially simplifying the system.

The plan of the paper is as follows.

In Section 2 we give the classical symmetries of the CNLS system (1.1). In Section 3 we discuss the automatic generation of the determining equations for nonclassical vector fields and the action of the classical symmetry group on these vector fields. The nonlinearity of the determining system leads to the necessity to treat cases. For the generic case, $\tau = 1$, Section 4 gives the solution of the determining equations and the corresponding reduced forms of the CNLS system. The determining equations for the case $\tau = 0$ are integrated in Section 5, and the corresponding ISCs are integrated in Section 6. The resulting reduced forms of the CNLS system are given in Section 7. We conclude with a discussion in Section 8.

2. Classical symmetries

In order to apply the Lie algorithm to obtain symmetries of (1.1), the equations comprising the system need to be analytic functions of the independent and dependent variables and derivative terms. One possibility is to write the system in terms of Φ, Ψ and their formal complex conjugates Φ^* and Ψ^* as

$$\begin{aligned}
&\mathrm{i}\Psi_t + \nabla^2\Psi + \left(\Psi\Psi^* + \Phi\Phi^*\right)\Psi = 0\,,\\
&-\mathrm{i}\Psi^*_t + \nabla^2\Psi^* + \left(\Psi\Psi^* + \Phi\Phi^*\right)\Psi^* = 0\,,\\
&\mathrm{i}\Phi_t + \nabla^2\Phi + \left(\Psi\Psi^* + \Phi\Phi^*\right)\Phi = 0\,,\\
&-\mathrm{i}\Phi^*_t + \nabla^2\Phi^* + \left(\Psi\Psi^* + \Phi\Phi^*\right)\Phi^* = 0\,.
\end{aligned}$$

In this article we chose an alternative representation, resulting from $\Psi = U + \mathrm{i}Q$, $\Psi^* = U - \mathrm{i}Q$, $\Phi = V + \mathrm{i}S$ and $\Phi^* = V - \mathrm{i}S$, in which the CNLS system is represented by the analytic system of four equations,

$$\begin{aligned}
\Delta_1 :&\quad U_t + \nabla^2 Q + \mathcal{R}^2 Q = 0\,,\\
\Delta_2 :&\quad Q_t - \nabla^2 U - \mathcal{R}^2 U = 0\,,\\
\Delta_3 :&\quad V_t + \nabla^2 S + \mathcal{R}^2 S = 0\,,\\
\Delta_4 :&\quad S_t - \nabla^2 V - \mathcal{R}^2 V = 0\,,
\end{aligned} \qquad (2.1)$$

with $\mathcal{R}^2 = U^2 + Q^2 + V^2 + S^2$. Classical Lie theory is an analytic theory and all quantities can be regarded as complex, including dependent and independent variables. Hence, the system (2.1) is more general than the system (1.1), which it yields as the special case where U, Q, V and S are real. One may take the view that U, Q, V and S should be real, given real independent variables. However, in this article we consider all variables to be complex. The reason is that for a particular application, the system to be studied may actually only be (1.1) after a change of variable, in which case the associated reality condition will be different.

To apply the classical method to the system (1.1), we consider the one-parameter Lie group of infinitesimal transformations in $(\boldsymbol{x}, \boldsymbol{U}) \equiv (x, y, z, t, U, Q, V, S)$ given by

$$\begin{aligned}
\widetilde{x} &= x + \epsilon\xi_1(\boldsymbol{x},\boldsymbol{U}) + \mathcal{O}(\epsilon^2), & \widetilde{U} &= U + \epsilon\phi_1(\boldsymbol{x},\boldsymbol{U}) + \mathcal{O}(\epsilon^2),\\
\widetilde{y} &= y + \epsilon\xi_2(\boldsymbol{x},\boldsymbol{U}) + \mathcal{O}(\epsilon^2), & \widetilde{Q} &= Q + \epsilon\phi_2(\boldsymbol{x},\boldsymbol{U}) + \mathcal{O}(\epsilon^2),\\
\widetilde{z} &= z + \epsilon\xi_3(\boldsymbol{x},\boldsymbol{U}) + \mathcal{O}(\epsilon^2), & \widetilde{V} &= V + \epsilon\phi_3(\boldsymbol{x},\boldsymbol{U}) + \mathcal{O}(\epsilon^2),\\
\widetilde{t} &= t + \epsilon\tau(\boldsymbol{x},\boldsymbol{U}) + \mathcal{O}(\epsilon^2), & \widetilde{S} &= S + \epsilon\phi_4(\boldsymbol{x},\boldsymbol{U}) + \mathcal{O}(\epsilon^2),
\end{aligned}$$

where ϵ is the group parameter. The corresponding classical vector field which generates the flow of the transformation is

$$\xi_1\partial_x + \xi_2\partial_y + \xi_3\partial_z + \tau\partial_t + \phi_1\partial_U + \phi_2\partial_Q + \phi_3\partial_V + \phi_4\partial_S .$$

The conditions that this is a classical symmetry vector field are obtained by requiring that the prolongation of the vector field leaves invariant the set

$$\mathcal{S}_\Delta \equiv \{U(\boldsymbol{x}), Q(\boldsymbol{x}), V(\boldsymbol{x}), S(\boldsymbol{x}) : \Delta_1 = 0, \Delta_2 = 0, \Delta_3 = 0, \Delta_4 = 0\} \tag{2.2}$$

of solutions of the system (2.1).

Lie's algorithm [10,59] yields a linear system of overdetermined PDE for the infinitesimals $\xi_i(\boldsymbol{x},\boldsymbol{U})$, $i = 1,2,3$, $\tau(\boldsymbol{x},\boldsymbol{U})$ and $\phi_j(\boldsymbol{x},\boldsymbol{U})$, $j = 1,2,3,4$. It has been implemented in many computer packages [40]. The calculations in this section were carried out in MAPLE V Release 5, on a 333 MHZ Pentium II PC running under Linux. When time derivatives in (2.1) were chosen as the terms to be substituted in the prolongation, Hickman's program `Symmetry` [41] automatically generated a raw classical determining system for the system (2.1). This system of 1888 linear PDE was generated in about 16 seconds using about 9 MB of RAM. When second order derivatives in x were chosen as the terms to be substituted in the prolongation, `Symmetry` automatically generated an equivalent raw determining system of 1144 PDE in about 8 seconds, using about 5 MB of RAM. Many programs, such as Hickman's, are not only able to produce raw (unsimplified) determining systems, but are also able to *carry out simplifications during the production of the system.* This can result in significantly smaller determining systems, and significantly less use of RAM. Indeed Alan Head's remarkable Mu-Math program Lie [39], using such strategies, was able to perform large symmetry calculations on vintage PCs with tiny memories in the late 1980's. The `rif` algorithm [73] reduced the classical determining system of 1144 equations to standard form in about 20 seconds and 4 MB of RAM. (The corresponding statistics for the equivalent system of 1888 equations were 40 seconds and 6 MB of RAM.) Application of the `initial_data` program [69] to the standard form obtained, showed that the Lie group of symmetries of the system (2.1) was 15-dimensional. Then the program `commutation_relations` [68,70] was used to determine the commutation relations of the symmetry algebra given in Fig. 1.

The above information was obtained without integrating the determining equations. We then explicitly obtained the generators as follows. The Taylor algorithm [72] was applied to the standard form with the given initial data to expand the infinitesimals ξ_1, ξ_2, ξ_3, τ, ϕ_1, ϕ_2, ϕ_3 and ϕ_4 to order two in Taylor series about an arbitrary point $(\boldsymbol{x}^0, \boldsymbol{U}^0)$. Expansion to order three yielded no additional terms. Since the expansion was done about an arbitrary point this meant that the expansion to order two gave the exact form of the infinitesimals. Choosing convenient (in fact mostly zero) values for the coordinates of the arbitrary point $(\boldsymbol{x}^0, \boldsymbol{U}^0)$ gave the following exact form of the infinitesimal generators:

$$\mathcal{L}_1 = \partial_x, \qquad \mathcal{L}_5 = \partial_y, \qquad \mathcal{L}_8 = \partial_z, \qquad \mathcal{L}_{10} = \partial_t,$$

$$\mathcal{L}_3 = z\partial_x - x\partial_z, \qquad \mathcal{L}_4 = y\partial_x - x\partial_y, \qquad \mathcal{L}_7 = z\partial_y - y\partial_z,$$

$$\mathcal{L}_2 = t\partial_x - \tfrac{1}{2}Qx\partial_U + \tfrac{1}{2}xU\partial_Q - \tfrac{1}{2}Sx\partial_V + \tfrac{1}{2}Vx\partial_S,$$

$[\mathcal{L}_1,\mathcal{L}_2]=\frac{1}{2}\mathcal{L}_{12}-\frac{1}{2}\mathcal{L}_{14}$, $[\mathcal{L}_1,\mathcal{L}_3]=-\mathcal{L}_8$, $[\mathcal{L}_1,\mathcal{L}_4]=-\mathcal{L}_5$, $[\mathcal{L}_1,\mathcal{L}_{11}]=-\mathcal{L}_1$,

$[\mathcal{L}_2,\mathcal{L}_3]=-\mathcal{L}_9$, $[\mathcal{L}_2,\mathcal{L}_4]=-\mathcal{L}_6$, $[\mathcal{L}_2,\mathcal{L}_{10}]=-\mathcal{L}_1$, $[\mathcal{L}_2,\mathcal{L}_{11}]=\mathcal{L}_2$,

$[\mathcal{L}_3,\mathcal{L}_4]=-\mathcal{L}_7$, $[\mathcal{L}_3,\mathcal{L}_7]=\mathcal{L}_4$, $[\mathcal{L}_3,\mathcal{L}_8]=-\mathcal{L}_1$, $[\mathcal{L}_3,\mathcal{L}_9]=-\mathcal{L}_2$,

$[\mathcal{L}_4,\mathcal{L}_5]=-\mathcal{L}_1$, $[\mathcal{L}_4,\mathcal{L}_6]=-\mathcal{L}_2$, $[\mathcal{L}_4,\mathcal{L}_7]=-\mathcal{L}_3$,

$[\mathcal{L}_5,\mathcal{L}_6]=\frac{1}{2}\mathcal{L}_{12}-\frac{1}{2}\mathcal{L}_{14}$, $[\mathcal{L}_5,\mathcal{L}_7]=-\mathcal{L}_8$, $[\mathcal{L}_5,\mathcal{L}_{11}]=-\mathcal{L}_5$,

$[\mathcal{L}_6,\mathcal{L}_7]=-\mathcal{L}_9$, $[\mathcal{L}_6,\mathcal{L}_{10}]=-\mathcal{L}_5$, $[\mathcal{L}_6,\mathcal{L}_{11}]=\mathcal{L}_6$,

$[\mathcal{L}_7,\mathcal{L}_8]=-\mathcal{L}_5$, $[\mathcal{L}_7,\mathcal{L}_9]=-\mathcal{L}_6$,

$[\mathcal{L}_8,\mathcal{L}_9]=\frac{1}{2}\mathcal{L}_{12}-\frac{1}{2}\mathcal{L}_{14}$, $[\mathcal{L}_8,\mathcal{L}_{11}]=-\mathcal{L}_8$,

$[\mathcal{L}_9,\mathcal{L}_{10}]=-\mathcal{L}_8$, $[\mathcal{L}_9,\mathcal{L}_{11}]=\mathcal{L}_9$,

$[\mathcal{L}_{10},\mathcal{L}_{11}]=-2\mathcal{L}_{10}$, $[\mathcal{L}_{12},\mathcal{L}_{13}]=\mathcal{L}_{15}$, $[\mathcal{L}_{12},\mathcal{L}_{15}]=-\mathcal{L}_{13}$,

$[\mathcal{L}_{13},\mathcal{L}_{14}]=-\mathcal{L}_{15}$, $[\mathcal{L}_{13},\mathcal{L}_{15}]=2\mathcal{L}_{12}+2\mathcal{L}_{14}$, $[\mathcal{L}_{14},\mathcal{L}_{15}]=-\mathcal{L}_{13}$.

Fig. 1. Nonvanishing commutation relations for CNLS system.

$$\begin{aligned}
\mathcal{L}_6 &= t\partial_y - \tfrac{1}{2}Qy\partial_U + \tfrac{1}{2}yU\partial_Q - \tfrac{1}{2}Sy\partial_V + \tfrac{1}{2}Vy\partial_S\,,\\
\mathcal{L}_9 &= t\partial_z - \tfrac{1}{2}Qz\partial_U + \tfrac{1}{2}zU\partial_Q - \tfrac{1}{2}Sz\partial_V + \tfrac{1}{2}Vz\partial_S\,,\\
\mathcal{L}_{11} &= -x\partial_x - y\partial_y - z\partial_z - 2t\partial_t + U\partial_U + Q\partial_Q + V\partial_V + S\partial_S\,,\\
\mathcal{L}_{12} &= -Q\partial_U + U\partial_Q\,, \mathcal{L}_{14} = S\partial_V - V\partial_S\,,\\
\mathcal{L}_{13} &= -V\partial_U - S\partial_Q + U\partial_V + Q\partial_S\,,\\
\mathcal{L}_{15} &= -S\partial_U + V\partial_Q - Q\partial_V + U\partial_S\,.
\end{aligned}$$

Our example calculation above is typical of calculations for classical symmetries of differential equations, especially for systems not containing parameters or arbitrary functions. Such calculations are now largely automatic, and as above, much of the information can be obtained without integration heuristics.

Having determined the infinitesimals, the symmetry variables for the reduced equations are found by solving the invariant surface conditions

$$\begin{aligned}
\psi_1 &\equiv \xi_1 U_x + \xi_2 U_y + \xi_3 U_z + \tau U_t - \phi_1 = 0\,,\\
\psi_2 &\equiv \xi_1 Q_x + \xi_2 Q_y + \xi_3 Q_z + \tau Q_t - \phi_2 = 0\,,\\
\psi_3 &\equiv \xi_1 V_x + \xi_2 V_y + \xi_3 V_z + \tau V_t - \phi_3 = 0\,,\\
\psi_4 &\equiv \xi_1 S_x + \xi_2 S_y + \xi_3 S_z + \tau S_t - \phi_4 = 0\,.
\end{aligned} \tag{2.3}$$

Several packages are now available which include heuristics to integrate the determining equations; an excellent survey article by Hereman gives the details of what is currently available [40] (see also [44], Chapter 13). For equations containing parameters, the package `diffgrob2` and `rif` can be used to find the special values of the parameters for which additional symmetries exist [53,69]. For equations containing arbitrary functions, so that a classification problem needs to be solved, there are several ways to proceed, but the most efficient appears to be via the use of the equivalence group. We refer the reader to [43,47,52].

As far as the authors are aware, the only computer algebra package which can integrate first order partial differential equations, that is, invariant surface conditions, in a form suitable for use in a reduction calculation, is the REDUCE package `crack`, written by Wolf and Brand [84].

3. Nonclassical reduction method

Bluman and Cole [9], in their study of symmetry reductions of the linear heat equation, proposed the so-called "nonclassical method of group-invariant solutions". This technique is also known as the "method of conditional symmetries" (cf. [46]). This method involves considerably more algebra and associated calculations than the linear classical Lie method since the associated determining equations are an overdetermined, *nonlinear* system. This has led some to question the feasibility of the method. Further, the associated vector fields arising from the nonclassical method do not form a vector space, still less a Lie algebra, since the invariant surface conditions (2.3) depend upon the particular reduction.

In the nonclassical method it is only required that the subset of $\mathcal{S}_\Delta$ given by

$$\mathcal{S}_{\Delta,\psi} = \{U(\boldsymbol{x}), Q(\boldsymbol{x}), V(\boldsymbol{x}), S(\boldsymbol{x}) : \Delta_1 = 0, \Delta_2 = 0, \Delta_3 = 0, \Delta_4 = 0,\ \psi_1 = 0,\ \psi_2 = 0,\ \psi_3 = 0,\ \psi_4 = 0\},$$

is invariant under the transformation generated by the vector field

$$\xi_1\partial_x + \xi_2\partial_y + \xi_3\partial_z + \tau\partial_t + \phi_1\partial_U + \phi_2\partial_Q + \phi_3\partial_V + \phi_4\partial_S .$$

This yields a nonlinear system of determining equations for the nonclassical infinitesimals $\xi_1, \xi_2, \dots, \phi_4$. Here $\mathcal{S}_\Delta$ is as defined in (2.2) and $\psi_1 = 0$, $\psi_2 = 0$, $\psi_3 = 0$ and $\psi_4 = 0$ are the invariant surface conditions (2.3).

As we showed in [23], the standard procedure for applying the nonclassical method (e.g., as described in [46]), can create difficulties, particularly when implemented in symbolic manipulation programs. These difficulties often arise for equations such as (1.1) which require the use of differential consequences of the invariant surface conditions (2.3). In [23] an algorithm was proposed for calculating the determining equations associated with the nonclassical method which avoids these difficulties, and we use that algorithm here. Further we showed how the MACSYMA package `symmgrp.max` [17], which was written to calculate the determining equations for the classical method, can be adapted to calculate the determining equations for the nonclassical method.

This algorithm proceeds by reducing the system (2.1) with respect to the invariant surface conditions (2.3), and then applying the classical Lie algorithm to the result. In order to minimise expression swell and keep the subsequent calculations tractable, one considers various cases separately. The generic case, Case 1, occurs when the component τ in the nonclassical vectorfield is nonzero: $\tau \neq 0$. Since the form of the ISC (2.3) is unchanged when multiplied by a nonzero function, there is no loss in setting $\tau = 1$, and the number of dependent variables is profitably reduced by one. Elimination of U_t, Q_t, V_t and S_t from (2.1) using (2.3) and then application of Lie's classical algorithm on the reduced system yields the determining equations for the nonclassical infinitesimals. The calculations for Case 1 and the reduction obtained is described in Section 4. If $\tau = 0$ there is no loss in setting one of the $\xi_i = 1$. Thus in Case 2 we choose $\tau = 0$ and $\xi_3 = 1$. All derivatives of functions with respect to z are eliminated from (2.1) using (2.3) until no z-derivatives appear. Lie's classical algorithm is then applied to the reduced system to yield the nonclassical determining system for Case 2. That system has two fewer dependent variables than the classical determining system and is smaller but more nonlinear than that for Case 1. The nonlinearity of the system means that there will be several solution branches caused by obtaining integrability conditions that factor, as well as the singular cases obtained by setting separants and leading coefficients to zero. Here we obtain solution branches $\xi_{1,x} \neq 0$ and $\xi_{1,x} = 0$. The solution branch $\xi_{1,x} = 0$ has three sub-branches given by $\xi_1 = 0$, $1 + \xi_1^2 \neq 0$ and $1 + \xi_1^2 = 0$. This last sub-branch splits into three cases, $\xi_2 = 0$, $1 - \xi_2^2 \neq 0$ and $1 - \xi_2^2 = 0$. The calculation and exposition of the various cases can be tedious, and does not always lead to genuinely different reductions. In this article we present the calculations for the generic subcase $\xi_{1,x} \neq 0$, and the singular subcase $\xi_{1,x} = 0$, $1 + \xi_1^2 \neq 0$ and $\xi_{2,x} \neq 0$. This leads to ten separate reduction families. For Case 2 reductions, the variable t is automatically one of the similarity variables.

Because permutations of the independent variables x, y and z leave (2.1) invariant, the case $\xi_1 = \xi_2 = 0$ is equivalent to Case 3, which is $\tau = \xi_3 = 0$ and $\xi_2 = 1$. We do not attempt this case here, nor do we consider Case 4, which would be $\tau = \xi_3 = \xi_2 = 0$ and $\xi_1 = 1$. The determining equations for these highly singular cases are typically intractable [58].

Two *classical* reduction solutions are said to be *equivalent* if one can be obtained from the other by the action of some member g in its classical symmetry group. Equivalently, the classical vector field $\widetilde{\boldsymbol{v}}$ of one reduction solution is obtained from the vector field of the other reduction solution $\boldsymbol{v}$ by the adjoint action of g: $\widetilde{\boldsymbol{v}} = g^{-1}\boldsymbol{v}g$. Thus we can simplify the presentation of classical reduction solutions, and their corresponding vector fields, by choosing just one representative from each equivalence (or so-called conjugacy) class. This use of the action of the classical group to remove unnecessary parameters is well established and has been shown to be effective in applications, for example by Patera, Winternitz and Zassenhaus [64].

In integrating the determining equations for nonclassical reductions, it has been observed on several occasions that certain constants of integration are, in the reduction equations obtained at the end of the process, removable by the action of the classical Lie group. The ability to remove such constants in the integration process of the determining equations themselves would lead to a considerable simplification of the calculations involved. Further, the integration of the ISCs would be much simpler. Vorob'ev [82], Section 4, gives a proof that the determining equations for nonclassical infinitesimals of a system inherits its classical symmetries. The proof uses the notations of the formal theory of differential systems. A simpler discussion can be found in [44], Section 11.3.4, where it is shown that a classical symmetry acting on a nonclassical vector field by its adjoint action yields a nonclassical vector field. We fruitfully apply this result in Section 4. Our understanding of the result is as follows.

The adjoint action of one vector field on another is given by

$$\mathrm{e}^{-a\boldsymbol{v}}\boldsymbol{w}\mathrm{e}^{a\boldsymbol{v}} = \boldsymbol{w} + [\boldsymbol{w},\boldsymbol{v}]a + [[\boldsymbol{w},\boldsymbol{v}],\boldsymbol{v}]\frac{a^2}{2!} + [[[\boldsymbol{w},\boldsymbol{v}],\boldsymbol{v}],\boldsymbol{v}]\frac{a^3}{3!} + \cdots.$$

Let $\boldsymbol{v}$ and $\boldsymbol{w}$ respectively be classical and nonclassical vector fields of a nondegenerate locally solvable system of PDE $\Delta = 0$. Let $\mathcal{F}$ be the (sufficiently) prolonged jet space locus of $\Delta = 0$. Let $\mathcal{I}$ be the (sufficiently prolonged) locus of the Invariant Surface Condition (ISC) of $\boldsymbol{w}$.

Since $\mathrm{e}^{\boldsymbol{v}}$ is a classical symmetry then locally $\mathrm{e}^{\boldsymbol{v}}\mathcal{F} = \mathcal{F}$. Since $\mathrm{e}^{\boldsymbol{w}}$ is a nonclassical vector field we have both $\mathrm{e}^{\boldsymbol{w}}\mathcal{I} = \mathcal{I}$ and $\mathrm{e}^{\boldsymbol{w}}(\mathcal{F}\cap\mathcal{I}) = \mathcal{F}\cap\mathcal{I}$. Let $\widehat{\mathcal{I}} = \mathrm{e}^{-\boldsymbol{v}}\mathcal{I}$. Then

$$\mathrm{e}^{-\boldsymbol{v}}\mathrm{e}^{\boldsymbol{w}}\mathrm{e}^{\boldsymbol{v}}\widehat{\mathcal{I}} = \mathrm{e}^{-\boldsymbol{v}}\mathrm{e}^{\boldsymbol{w}}\mathcal{I} = \mathrm{e}^{-\boldsymbol{v}}\mathcal{I} = \widehat{\mathcal{I}}$$

so $\widehat{\mathcal{I}}$ is the ISC for $\mathrm{e}^{-\boldsymbol{v}}\mathrm{e}^{\boldsymbol{w}}\mathrm{e}^{\boldsymbol{v}}$. Also

$$\mathrm{e}^{-\boldsymbol{v}}\mathrm{e}^{\boldsymbol{w}}\mathrm{e}^{\boldsymbol{v}}\left(\mathcal{F}\cap\widehat{\mathcal{I}}\right) = \mathrm{e}^{-\boldsymbol{v}}\mathrm{e}^{\boldsymbol{w}}\left(\mathrm{e}^{\boldsymbol{v}}\mathcal{F}\cap\mathrm{e}^{\boldsymbol{v}}\widehat{\mathcal{I}}\right) = \mathrm{e}^{-\boldsymbol{v}}\mathrm{e}^{\boldsymbol{w}}(\mathcal{F}\cap\mathcal{I}).$$

Since $\mathrm{e}^{\boldsymbol{w}}(\mathcal{F}\cap\mathcal{I}) = \mathcal{F}\cap\mathcal{I}$, we have

$$\mathrm{e}^{-\boldsymbol{v}}\mathrm{e}^{\boldsymbol{w}}\mathrm{e}^{\boldsymbol{v}}\left(\mathcal{F}\cap\widehat{\mathcal{I}}\right) = \mathrm{e}^{-\boldsymbol{v}}(\mathcal{F}\cap\mathcal{I}) = \mathrm{e}^{-\boldsymbol{v}}\mathcal{F}\cap\mathrm{e}^{-\boldsymbol{v}}\mathcal{I} = \mathcal{F}\cap\widehat{\mathcal{I}}.$$

Consequently both $\mathrm{e}^{-\boldsymbol{v}}\mathrm{e}^{\boldsymbol{w}}\mathrm{e}^{\boldsymbol{v}}(\mathcal{F}\cap\widehat{\mathcal{I}}) = \mathcal{F}\cap\widehat{\mathcal{I}}$ and $\mathrm{e}^{-\boldsymbol{v}}\mathrm{e}^{\boldsymbol{w}}\mathrm{e}^{\boldsymbol{v}}\widehat{\mathcal{I}} = \widehat{\mathcal{I}}$, so $\mathrm{e}^{-\boldsymbol{v}}\mathrm{e}^{\boldsymbol{w}}\mathrm{e}^{\boldsymbol{v}}$ is a nonclassical vector field of $\Delta = 0$.

4. Nonclassical reductions, Case 1: $\tau = 1$

The nonclassical determining equations for the case $\tau = 1$ were automatically generated using the MAPLE program [41] using the algorithm described in [23]. As in Section 2, the calculations in this section were carried out in MAPLE V Release 5, on a 333 MHZ Pentium II PC running under Linux. Hickman's MAPLE program `Symmetry` [41] automatically generated a raw nonlinear nonclassical determining system of 856 PDE in about 8 seconds using about 5 MB of RAM.

An incomplete run of the `rif` algorithm [73] showed that the system contained a simple subsystem of single term equations. Specifically this subsystem consisted of a system with all second order partial derivatives of ϕ_1, ϕ_2, ϕ_3 and ϕ_4 with respect to x, y, z and t, equal to zero. Thus by simple integration

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{pmatrix} \begin{pmatrix} U \\ Q \\ V \\ S \end{pmatrix} + \begin{pmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{pmatrix}, \tag{4.1}$$

where $G_{jk} = G_{jk}(x, y, z, t)$ and $G_j = G_j(x, y, z, t)$ are functions of integration.

Wittkopf's `rif` algorithm was then applied to the system resulting from substitution of (4.1) into the determining system. Using its automatic case splitting option, in about 120 seconds and 4 MB of RAM, his algorithm yielded 4 cases. Each case was in `rif`-form and one was inconsistent. The `initial_data` algorithm was applied to the remaining three cases. The exact forms of the classical symmetries were substituted into the `rif`-forms to determine the number of parameters in the subclass of classical symmetries which satisfied each case. If this was the same as the number calculated by the `initial_data` algorithm, then the case was rejected as not yielding a genuinely nonclassical vector field. On this basis only one of the three consistent cases was identified as being genuinely nonclassical. For that case the `initial_data` algorithm yielded 9 parameters, whereas the subclass of classical solutions only contained 8 parameters. The `rif`-form for this case is given in Fig. 2. We mention that it is possible to determine whether a case is genuinely nonclassical without integration. Instead of substitution of the exact classical solutions used above, reduction of the `rif`-form of the nonclassical system with respect to the `rif`-form of the classical determining system is used.

The leading nonlinear subsystem in the `rif`-form consisted of the single equation $G_{41}^2 + G_{42}^2 = 0$ together with the pivot condition $\Lambda = G_{41} + G_{32} \neq 0$. The last condition is equivalent to $G_{41} \neq 0$, since $G_{32} = G_{41}$ occurs in the `rif`-form. The ranking used in the reduction was graded first by total order of derivative, then lexicographically by derivative ($\partial_x \succ \partial_y \succ \partial_z \succ \partial_t \succ \partial_U \succ \partial_Q \succ \partial_V \succ \partial_S$) and finally lexicographically on dependent variable ($\xi_1 \succ \xi_2 \succ \xi_3 \succ \tau \succ G_{11} \succ G_{12} \succ G_{13} \succ G_{14} \succ G_{21} \succ \cdots \succ G_{24} \succ G_{31} \succ \cdots \succ G_{34} \succ G_{41} \succ \cdots \succ G_{44} \succ G_1 \succ G_2 \succ G_3 \succ G_4$).

4.1. Integration of the determining equations

There are two cases: $G_{41} - \mathrm{i}G_{42} = 0$ and $G_{41} + \mathrm{i}G_{42} = 0$. Since the solution set of the determining system depends on finitely many parameters, we know that if a ranking of lexicographic type [16,75] is used, then the `rif`-form is guaranteed to contain a parameterized ODE [54,71]. Recursively applying this procedure reduces the integration of the system to the integration of ODEs. We used MAPLE's dsolve, to solve these ODEs. This easily led to the following exact solution for the case $G_{41} - \mathrm{i}G_{42} = 0$:

$$\xi_1 = \frac{2b_1 t - x - \mathrm{i}b_6}{b_5 - 2t}, \qquad \xi_2 = \frac{2b_2 t - y - \mathrm{i}b_7}{b_5 - 2t},$$

$$\xi_3 = \frac{2b_3 t - z - \mathrm{i}b_8}{b_5 - 2t}, \qquad \tau = 1,$$

$$\phi_1 = \frac{U - Q(b_1 x + b_2 y + b_3 z + b_4)}{b_5 - 2t} + (V - \mathrm{i}S)\,\mathrm{e}^{\mathrm{i}H},$$

$$\phi_2 = \frac{Q + U(b_1 x + b_2 y + b_3 z + b_4)}{b_5 - 2t} - \mathrm{i}(V - \mathrm{i}S)\,\mathrm{e}^{\mathrm{i}H},$$

Leading Linear PDE :

$\partial_{tt}\xi_1 = 4G_{44}\partial_t\xi_1$, $\partial_{tt}\xi_2 = 4G_{44}\partial_t\xi_2$, $\partial_{tt}\xi_3 = 4G_{44}\partial_t\xi_3$,

$\partial_x\xi_1 = -G_{44}$, $\partial_x\xi_2 = 0$, $\partial_x\xi_3 = 0$, $\partial_x G_{41} = -\xi_1 G_{42}$,

$\partial_x G_{42} = \xi_1 G_{41}$, $\partial_x G_{43} = \frac{1}{2}\partial_t\xi_1 - \xi_1 G_{44}$, $\partial_x G_{44} = 0$,

$\partial_y\xi_1 = 0$, $\partial_y\xi_2 = -G_{44}$, $\partial_y\xi_3 = 0$, $\partial_y G_{41} = -\xi_2 G_{42}$, $\partial_y G_{42} = \xi_2 G_{41}$,

$\partial_y G_{43} = \frac{1}{2}\partial_t\xi_2 - \xi_2 G_{44}$, $\partial_y G_{44} = 0$,

$\partial_z\xi_1 = 0$, $\partial_z\xi_2 = 0$, $\partial_z\xi_3 = -G_{44}$, $\partial_z G_{41} = -\xi_3 G_{42}$,

$\partial_z G_{42} = \xi_3 G_{41}$, $\partial_z G_{43} = \frac{1}{2}\partial_t\xi_3 - \xi_3 G_{44}$, $\partial_z G_{44} = 0$,

$\partial_t G_{41} = (\xi_1^2 + \xi_2^2 + \xi_3^2 - 2G_{43})G_{42} + 3G_{41}G_{44}$,

$\partial_t G_{42} = 3G_{42}G_{44} + (2G_{43} - \xi_1^2 - \xi_2^2 - \xi_3^2)G_{41}$,

$\partial_t G_{43} = 2G_{43}G_{44}$, $\partial_t G_{44} = 2G_{44}^2$,

$\partial_U\xi_1 = 0$, $\partial_U\xi_2 = 0$, $\partial_U\xi_3 = 0$, $\partial_U G_{41} = 0$, $\partial_U G_{42} = 0$, $\partial_U G_{43} = 0$, $\partial_U G_{44} = 0$,

$\partial_Q\xi_1 = 0$, $\partial_Q\xi_2 = 0$, $\partial_Q\xi_3 = 0$, $\partial_Q G_{41} = 0$, $\partial_Q G_{42} = 0$, $\partial_Q G_{43} = 0$, $\partial_Q G_{44} = 0$,

$\partial_V\xi_1 = 0$, $\partial_V\xi_2 = 0$, $\partial_V\xi_3 = 0$, $\partial_V G_{41} = 0$, $\partial_V G_{42} = 0$, $\partial_V G_{43} = 0$, $\partial_V G_{44} = 0$,

$\partial_S\xi_1 = 0$, $\partial_S\xi_2 = 0$, $\partial_S\xi_3 = 0$, $\partial_S G_{41} = 0$, $\partial_S G_{42} = 0$, $\partial_S G_{43} = 0$, $\partial_S G_{44} = 0$,

$\tau = 1$, $G_{11} = G_{44}$, $G_{12} = -G_{43}$, $G_{13} = G_{42}$, $G_{14} = -G_{41}$, $G_{21} = G_{43}$, $G_{22} = G_{44}$,

$G_{23} = -G_{41}$, $G_{24} = -G_{42}$, $G_{31} = -G_{42}$, $G_{32} = G_{41}$, $G_{33} = G_{44}$, $G_{34} = -G_{43}$,

$G_1 = 0$, $G_2 = 0$, $G_3 = 0$, $G_4 = 0$

Leading Nonlinear PDE : $G_{41}^2 + G_{42}^2 = 0$

Pivot Conditions : $(G_{41} + G_{32} \neq 0)$

Fig. 2. Reduced Involutive Form, Nonclassical Case 1 (Section 4).

$$\phi_3 = \frac{V - S(b_1x + b_2y + b_3z + b_4)}{b_5 - 2t} - (U - \mathrm{i}Q)\,\mathrm{e}^{\mathrm{i}H},$$
$$\phi_4 = \frac{S + V(b_1x + b_2y + b_3z + b_4)}{b_5 - 2t} + \mathrm{i}(U - \mathrm{i}Q)\,\mathrm{e}^{\mathrm{i}H},$$

where

$$H = \frac{(b_5 - 2t)(x\xi_1 + y\xi_2 + z\xi_3) + \frac{1}{2}(x^2 + y^2 + z^2) - \mathrm{i}G}{b_5 - 2t},$$
$$G = A(b_5 - 2t) + B - C(b_5 - 2t)\log(b_5 - 2t),$$
$$A = b_9 - \mathrm{i}(b_1^2 + b_2^2 + b_3^2)t,$$
$$B = \tfrac{1}{2}\mathrm{i}\left[(b_6 + \mathrm{i}b_1b_5)^2 + (b_7 + \mathrm{i}b_2b_5)^2 + (b_8 + \mathrm{i}b_3b_5)^2\right],$$
$$C = \mathrm{i}b_4 + \tfrac{3}{2} + b_1(b_6 + \mathrm{i}b_1b_5) + b_2(b_7 + \mathrm{i}b_2b_5) + b_3(b_8 + \mathrm{i}b_3b_5),$$

and B, C, $b_1, b_2, \ldots, b_9$ are complex constants. The case $G_{41} + \mathrm{i}G_{42} = 0$ is treated similarly.

4.2. Simplification of the vector field and integration of the characteristic equations

For convenience we multiply the vector field whose components are given above by $(b_5 - 2t)$. This yields an equivalent nonclassical vector field,

$$w = (b_5 - 2t)\left(\xi_1\partial_x + \xi_2\partial_y + \xi_3\partial_z + \tau\partial_t + \phi_1\partial_U + \phi_2\partial_Q + \phi_3\partial_V + \phi_4\partial_S\right),$$

where the infinitesimals ξ_1, etc., are given above. Using the explicit expressions for the classical Lie symmetries $\mathcal{L}_k$ in Section 2, the vector field w can be written in the form

$$w = u + n,$$

where u is a classical vector field and

$$u = \mathcal{L}_{11} - \mathrm{i}b_6\mathcal{L}_1 - \mathrm{i}b_7\mathcal{L}_5 - \mathrm{i}b_8\mathcal{L}_8 + b_5\mathcal{L}_{10} + 2b_1\mathcal{L}_2 + 2b_2\mathcal{L}_6 + 2b_3\mathcal{L}_9 + b_4(\mathcal{L}_{12} - \mathcal{L}_{14}),$$
$$n = \exp(\mathrm{i}H)\mathcal{N},$$
$$\mathcal{N} = (V - \mathrm{i}S)(\partial_U - \mathrm{i}\partial_Q) - (U - \mathrm{i}Q)(\partial_V - \mathrm{i}\partial_S).$$

We know from Section 3 that the action of the classical group on w will produce another nonclassical vector field. We will use the classical group action to remove parameters and simplify w. Since w has a large classical part u, which depends on most of the parameters, we first simplified u with respect to the classical group.

Theorem 1. Under the adjoint action of the classical group of the CNLS system the vector field u is equivalent to the vector field $\mathcal{L}_{11}$.

Thus effectively the troublesome parameters $b_1, b_2, \ldots, b_8$ can be set to zero.

Proof. To prove this result one applies appropriate adjoint actions using the iterated commutator formula and the commutation relations in Fig. 1. Application of the adjoint action of time translation effectively sets b_5 to zero. Then application of spatial translations effectively sets b_6, b_7 and b_8 to zero. A slightly more complicated calculation based on applying the adjoint action of $\mathcal{L}_2$ puts $b_1 = 0$. Similarly, application of the adjoint actions of $\mathcal{L}_6$ and $\mathcal{L}_9$, respectively, enables us to take $b_2 = 0$ and $b_3 = 0$. Now u has the form $\mathcal{L}_{11} + b_4(\mathcal{L}_{12} - \mathcal{L}_{14})$. Application of the action of $\mathcal{L}_2$ transforms u to the form $\mathcal{L}_{11} + b_4(\mathcal{L}_{12} - \mathcal{L}_{14}) + \alpha\mathcal{L}_2$, for some nonzero α. From Fig. 1, $[\mathcal{L}_2, \mathcal{L}_1] = -\frac{1}{2}(\mathcal{L}_{12} - \mathcal{L}_{14})$, and $[[\mathcal{L}_2, \mathcal{L}_1], \mathcal{L}_1] = 0$. This implies that the application of the appropriate action of $\mathcal{L}_1$ to u can transform it to $\mathcal{L}_{11} + \alpha\mathcal{L}_2 + \beta\mathcal{L}_1$. The terms in $\mathcal{L}_1$, $\mathcal{L}_2$ can be removed by applying spatial translations, and this completes the proof. □

The above calculations were easily and quickly accomplished by hand. Application of the the same adjoint actions to the full nonclassical vector field w yields the following result:

Theorem 2. Under the adjoint action of the classical group of the CNLS system the vector field $w = u + n$ is equivalent to the vector field $\widetilde{w} = \mathcal{L}_{11} + \widehat{n}$ where $\widehat{n} = at^{-1/2}\exp\left\{\mathrm{i}(x^2 + y^2 + z^2)/(4t)\right\}\mathcal{N}$.

To prove this result the same classical group actions are applied in the same sequence as in the proof of Theorem 1. Simple computations show that there are functions f_j such that $[\mathcal{N}, \mathcal{L}_j] = f_j\mathcal{N}$, for $j = 1, 2, \ldots, 15$. From Section 3, we know that nonclassical vector fields are mapped to nonclassical vector fields under the action of the classical group. Together with $[\mathcal{N}, \mathcal{L}_j] = f_j\mathcal{N}$ this fact enables the proof of Theorem 2 to be easily completed without explicitly computing the exact form of the adjoint actions, or the f_j.

Effectively, Theorem 2 means that we can set $b_1, \ldots, b_8$ to zero. Redefinition of b_9 yields the constant a in the theorem. Note that $\widetilde{w}$ is a classical vector field when $a = 0$.

Integrating the characteristic equations for $\widetilde{w}$ yields the similarity variables,

$$z_1 = x/t^{1/2}, \qquad z_2 = y/t^{1/2}, \qquad z_3 = z/t^{1/2},$$

and the form of the reduced solutions,

$$\begin{aligned}
\Psi(x,y,z,t) &= t^{-1/2}B_3(z_1,z_2,z_3) - 4at^{-1}\exp\left(\tfrac{1}{4}\mathrm{i}\zeta^2\right)B_2(z_1,z_2,z_3),\\
\Psi^*(x,y,z,t) &= t^{-1/2}B_1(z_1,z_2,z_3),\\
\Phi(x,y,z,t) &= t^{-1/2}B_4(z_1,z_2,z_3) + 4at^{-1}\exp\left(\tfrac{1}{4}\mathrm{i}\zeta^2\right)B_1(z_1,z_2,z_3),\\
\Phi^*(x,y,z,t) &= t^{-1/2}B_2(z_1,z_2,z_3),
\end{aligned}$$

where $\zeta^2 \equiv z_1^2 + z_2^2 + z_3^2$, $\Phi = U + \mathrm{i}Q$, $\Phi^* = U - \mathrm{i}Q$, $\Psi = V + \mathrm{i}S$ and $\Psi^* = V - \mathrm{i}S$.

The above results correspond to the case $G_{41} - \mathrm{i}G_{42} = 0$. The case $G_{41} + \mathrm{i}G_{42} = 0$ is obtained from the above results by applying the symmetry of the system (2.1) under conjugation, $\Psi \mapsto \Psi^*$, $\Psi^* \mapsto \Psi$, $\Phi \mapsto \Phi^*$ and $\Phi^* \mapsto \Phi$.

Finally substitution of the above expressions into the CNLS system yields the reduced form,

$$\begin{aligned}
\mathcal{K}B_1 &= -\tfrac{1}{2}\mathrm{i}\,(1+\mathcal{M})\,B_1, \qquad & \mathcal{K}B_2 &= -\tfrac{1}{2}\mathrm{i}\,(1+\mathcal{M})\,B_2,\\
\mathcal{K}B_3 &= \tfrac{1}{2}\mathrm{i}\,(1+\mathcal{M})\,B_3, \qquad & \mathcal{K}B_4 &= \tfrac{1}{2}\mathrm{i}\,(1+\mathcal{M})\,B_4,
\end{aligned}$$

where $\mathcal{K} = \partial_{z_1}^2 + \partial_{z_2}^2 + \partial_{z_3}^2 + B_1B_3 + B_2B_4$ and $\mathcal{M} = z_1\partial_{z_1} + z_1\partial_{z_1} + z_1\partial_{z_1}$. Note that a does not appear explicitly in the reduced equations so the functions B_j for $a \neq 0$ are the same as those for the classical case $a = 0$. Thus the form of the reduced solutions for Ψ, Ψ^*, Φ and Φ^* above can be regarded as a map from classical solutions to nonclassical solutions.

Consider the obvious reality condition for the CNLS system where $a = 0$, and * denotes complex conjugation and x, y, z, t are real. Then $B_3^* = B_1$ and $B_4^* = B_2$. The case $a \neq 0$ is not consistent with this obvious reality condition. However, the nonclassical reductions above for $a \neq 0$ are still potentially of use in applications where the CNLS system does not have the obvious reality condition (such as after a change of variables).

The $N + 1$-dimensional CNLS system admits the classical scaling symmetry

$$\mathcal{L}_{scal} = -\sum_{j=1}^{N} x_j\partial_{x_j} - 2t\partial_t + U\partial_U + Q\partial_Q + V\partial_V + S\partial_S.$$

We have checked that the vector field

$$\mathcal{L}_{scal} + at^{(2-N)/2}\exp\left\{\frac{\mathrm{i}}{4t}\sum_{j=1}^{N} x_j^2\right\}\mathcal{N}$$

is a genuine nonclassical vector field for the $N + 1$-dimensional CNLS system, for $N = 1, 2, 3, 4$. When $N = 3$ it yields the nonclassical vector field studied in this section. We conjecture that it is a genuine nonclassical vector field for general N. For $N = 1, 3, 4$, the corresponding reduced solutions take the same form as above, except that t^{-1} in the expressions for Φ and Ψ is replaced with $t^{(1-N)/2}/(N-2)$.

5. Nonclassical infinitesimals, Case 2: $\tau = 0$, $\xi_3 \neq 0$

Setting $\xi_3 = 1$, without loss of generality, the invariant surface conditions (2.3) simplify to

$$\xi_1 U_x + \xi_2 U_y + U_z = \phi_1 ,$$
$$\xi_1 Q_x + \xi_2 Q_y + Q_z = \phi_2 ,$$
$$\xi_1 V_x + \xi_2 V_y + V_z = \phi_3 ,$$
$$\xi_1 S_x + \xi_2 S_y + S_z = \phi_4 . \tag{5.1}$$

Analogous to the procedure in the generic case, we use (5.1) to eliminate all z-derivatives. Thus U_{zz}, Q_{zz}, V_{zz} and S_{zz}, and other z derivatives are eliminated from (1.1), and the classical Lie algorithm is applied to the resulting system.

The calculations in Sections 5, 6 and 7 were carried out using MAPLE V, Release 4, on a Silicon Graphics Workstation operating under IRIX Release 5.3, with 48 MB of RAM and a 33 MHZ IP12 Processor. The calculations were done in an interactive fashion, making extensive use of `diffgrob2`. Most terminated in seconds or minutes, and required several MB of RAM. The MACSYMA package `symmgrp.max` [17] was used to create the determining system. It contained 238 determining equations in 6 dependent and 8 independent variables. As in the generic case, a subsystem of single term equations was simply integrated to give $\phi_1, \dots, \phi_4$ as in (4.1). A second subsystem of two term linear equations were solved by `diffgrob2` [50,51]. The system (4.1) immediately simplified to

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} 0 & G_{12} & G_{13} & G_{14} \\ -G_{12} & 0 & -G_{14} & G_{13} \\ -G_{13} & G_{14} & 0 & G_{34} \\ -G_{14} & -G_{13} & -G_{34} & 0 \end{pmatrix} \begin{pmatrix} U \\ Q \\ V \\ S \end{pmatrix},$$

where $G_{ij} = G_{ij}(x, y, z, t)$.

5.1. Generic case $\xi_{1,x} \neq 0$

Inserting (5.1) into the remaining equations yielded a number of nonlinear equations for ξ_1 and ξ_2. Calculating integrability conditions on this subsystem yielded a system, given $\xi_{1,x} \neq 0$, which was solved easily to give

$$\xi_1 = \frac{z + k_3(t) y + k_4(t)}{k_1(t) y + k_2(t) - x}, \qquad \xi_2 = \frac{-k_1(t) z - k_3(t) x + k_5(t)}{k_1(t) y + k_2(t) - x}.$$

Substituting these into the remaining equations yielded several first order linear equations for the G_{ij}, which were solved using the method of characteristics to yield

$$G_{12} = \frac{w_{12}(\xi_1, y, t)}{k_1(t) y + k_2(t) - x} + \frac{1}{2}\left(\frac{dk_3}{dt} y + \frac{dk_4}{dt} - \xi_1\left(\frac{dk_1}{dt} y + \frac{dk_2}{dt}\right)\right),$$
$$G_{13} = \frac{w_{13}(\xi_1, y, t)}{k_1(t) y + k_2(t) - x}, \qquad G_{14} = \frac{w_{14}(\xi_1, y, t)}{k_1(t) y + k_2(t) - x},$$
$$G_{34} = \frac{w_{34}(\xi_1, y, t)}{k_1(t) y + k_2(t) - x} + \frac{1}{2}\left(\frac{dk_3}{dt} y + \frac{dk_4}{dt} - \xi_1\left(\frac{dk_1}{dt} y + \frac{dk_2}{dt}\right)\right),$$

where the w_{ij} are arbitrary functions of their arguments. Calculating further integrability conditions yielded

$$\frac{dk_1}{dt} = 0, \quad \frac{dk_3}{dt} = 0, \quad \frac{d^2 k_2}{dt^2} = 0, \quad \frac{d^2 k_4}{dt^2} = 0,$$
$$k_5 = k_1 k_4 + k_3 k_2, \qquad w_{ij,y} = w_{ij,t} = 0,$$

and four other nonlinear equations in the w_{ij}. Integration of this system gave

$$w_{34}^2 - w_{12}^2 = c_1\,, \qquad w_{12}^2 + w_{13}^2 + w_{14}^2 = c_2\,,$$
$$w_{13}(w_{34}+w_{12}) = c_3\,, \qquad w_{14}(w_{34}+w_{12}) = c_4\,, \tag{5.2}$$

where the c_i are constants of integration. Applying MAPLE's Gröbner basis package to (5.2) yielded a quartic for w_{34} with coefficients in terms of the c_j's and then w_{12}, w_{13}, w_{14} in terms of w_{34}. Thus the w_{ij} are constants unless the quartic is identically zero. Performing another Gröbner basis calculation on the coefficients of the quartic showed that $c_1 = c_2 = 0, c_3^2 + c_4^2 = 0$ is the exceptional case. Thus we obtain two cases with $\xi_{1,x} \neq 0$:

Case I: The w_{ij} are all constant, and

Case II: $w_{34} = -w_{12}$ and $w_{12}^2 + w_{13}^2 + w_{14}^2 = 0$.

In Case I, there are no further equations to solve. We now restrict ourselves to Case II. A differential Gröbner basis calculation with the remaining equations shows that the w_{ij} are proportional to each other and satisfy the following ordinary differential equation:

$$\left[(1+k_1^2)\xi_1^2 - 2k_1k_3\xi_1 + 1 + k_3^2\right]\frac{\mathrm{d}^2 w}{\mathrm{d}\xi_1^2} + \left[2(1+k_1^2)\xi_1 - 2k_1k_3\right]\frac{\mathrm{d}w}{\mathrm{d}\xi_1} = 0\,. \tag{5.3}$$

There are four solution branches of (5.3) depending on the values of k_1 and k_3.

Case IIa: If $1 + k_1^2 + k_3^2 \neq 0$ and $1 + k_1^2 \neq 0$, then

$$w = \frac{\alpha_1}{\sqrt{1+k_1^2+k_3^2}}\arctan\left(\frac{(1+k_1^2)\xi_1 - k_1k_3}{\sqrt{1+k_1^2+k_3^2}}\right) + \alpha_2\,. \tag{5.4}$$

Case IIb: If $1 + k_1^2 + k_3^2 = 0$ and $k_1 \neq \pm\mathrm{i}$, then

$$w = \frac{\alpha_1}{(1+k_1^2)\left(\xi_1 + k_1/k_3\right)} + \alpha_2\,. \tag{5.5}$$

Case IIc: If $k_1 = \pm\mathrm{i}$ and $k_3 \neq 0$, then

$$w = \alpha_1 \ln\left(\mp 2\mathrm{i}k_3\xi_1 + 1 + k_3^2\right) + \alpha_2\,. \tag{5.6}$$

Case IId: If $k_1 = \pm\mathrm{i}$ and $k_3 = 0$, then

$$w = \alpha_1\xi_1 + \alpha_2\,. \tag{5.7}$$

5.2. *Singular case: $\xi_{1,x} = 0,\ 1 + \xi_1^2 \neq 0,\ \xi_{2,x} \neq 0$*

Inserting (5.1) and $\xi_{1,x} = 0$ into the determining equations yields $\xi_1 = \xi_1(t)$, and a subsystem for ξ_2. Assuming that $\xi_{2,x} \neq 0$, then this subsystem has solution

$$\xi_2 = \frac{\xi_1(t)x + z + h(t)}{g(t) - y}\,, \tag{5.8}$$

where $h(t)$ and $g(t)$ are arbitrary functions. A set of linear first order equations for the G_{ij} are also obtained. An integrability condition of that system implies that $\xi_{1,t} = 0$. The system can then be solved by the method of characteristics to yield

$$G_{12} = \frac{w_{12}(\xi_2,t)}{g(t)-y} + \frac{1}{2}\left(\frac{\mathrm{d}h}{\mathrm{d}t} - \xi_2\frac{\mathrm{d}g}{\mathrm{d}t}y\right), \qquad G_{13} = \frac{w_{13}(\xi_2,t)}{g(t)-y}\,,$$
$$G_{14} = \frac{w_{14}(\xi_2,t)}{g(t)-y}\,, \qquad G_{34} = \frac{w_{34}(\xi_2,t)}{g(t)-y} + \frac{1}{2}\left(\frac{\mathrm{d}h}{\mathrm{d}t} - \xi_2\frac{\mathrm{d}g}{\mathrm{d}t}y\right),$$

where the w_{ij} are arbitrary functions of their arguments. Substituting these into the remaining equations yields equations of the type

$$w_{14,t}(g(t)-y)^2 = (w_{14}(w_{12}+w_{34}))_{\xi_2}\,,$$

where the subscripts denote partial derivatives. Using this equation and the form for ξ_2 above shows that $w_{14,t}=0$. The same argument applied to the remaining w_{ij} shows that they depend only on ξ_2. Using this fact in the remaining equations yields

$$\frac{d^2g}{dt^2}=0\,,\qquad \frac{d^2h}{dt^2}=0$$

and a set of four equations which are easily integrated to yield the same system (5.2) as obtained before. Following the same reasoning as in Section 5.1 yields that the w_{ij} are all constants, unless $w_{12}+w_{34}=0$ and $w_{12}^2+w_{13}^2+w_{14}^2=0$. Using $w_{12}+w_{34}=0$ in the previous equations yields

$$\begin{aligned}
&w_{12}''(1+\xi_1^2+\xi_2^2)+2w_{12}'\xi_2+2w_{14}w_{13}'-2w_{13}w_{14}'=0\,,\\
&w_{13}''(1+\xi_1^2+\xi_2^2)+2w_{13}'\xi_2+2w_{12}w_{14}'-2w_{14}w_{12}'=0\,,\\
&w_{14}''(1+\xi_1^2+\xi_2^2)+2w_{14}'\xi_2+2w_{13}w_{12}'-2w_{12}w_{13}'=0\,,
\end{aligned}$$

where $'=d/d\xi_2$. One obvious nonconstant solution is that the w_{1j} are all proportional, so that

$$w_{1j}=\gamma_j\arctan\left(\frac{\xi_2}{\sqrt{1+\xi_1^2}}\right)\,, \tag{5.9}$$

together with $\gamma_1^2+\gamma_2^2+\gamma_3^2=0$. (Recall all variables are complex.) Attempting to show that this is the only nonconstant solution for the w_{1j} using integrability conditions leads to tremendous expression swell. Instead, we write (w_{12},w_{13},w_{14}) as a vector $\boldsymbol{v}$ and the system in terms of the vectors $\boldsymbol{v}''$, $\boldsymbol{v}'$ and $\boldsymbol{v}'\times\boldsymbol{v}$. Then $|\boldsymbol{v}|^2=0$, implies that $\boldsymbol{v}'\cdot\boldsymbol{v}=0$. Using the identity $(\boldsymbol{v}'\times\boldsymbol{v})\times\boldsymbol{v}=-\boldsymbol{v}'|\boldsymbol{v}|^2+(\boldsymbol{v}'\cdot\boldsymbol{v})\boldsymbol{v}$ shows that the system can be written as $\left((1+\xi_1^2+\xi_2^2)\boldsymbol{v}'\times\boldsymbol{v}\right)'=0$. The result that the w_{1j} are proportional is easily obtained.

In summary, for $1+\xi_1^2\neq 0$ and $\xi_{2,x}\neq 0$ we obtain two further cases:

Case III: the w_{ij} are all constant, and

Case IV: $w_{34}=-w_{12}$, $w_{12}^2+w_{13}^2+w_{14}^2=0$ and the w_{ij} are of the form (5.9).

The case $\xi_{2,x}=0$ implies $\xi_1=0$. This a subcase of Case 3 which is not attempted in this paper (see the discussion in Section 3).

6. Integration of the invariant surface conditions, Case 2, $\tau = 0$

6.1. Generic case $\xi_{1,x}\neq 0$

Using the infinitesimals obtained in Section 5, the invariant surface conditions can be written in the form

$$\begin{pmatrix} U_x & U_y & U_z & U_t\\ Q_x & Q_y & Q_z & Q_t\\ V_x & V_y & V_z & V_t\\ S_x & S_y & S_z & S_t\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\\ 1\\ 0\end{pmatrix}=\frac{w(\xi_1)}{k_1y-x+k_2}\begin{pmatrix}0 & \gamma_1 & \gamma_2 & \gamma_3\\ -\gamma_1 & 0 & -\gamma_3 & \gamma_2\\ -\gamma_2 & \gamma_3 & 0 & \gamma_4\\ -\gamma_3 & -\gamma_2 & -\gamma_4 & 0\end{pmatrix}\begin{pmatrix}U\\ Q\\ V\\ S\end{pmatrix}$$

$$+\frac{\kappa_4-\xi_1\kappa_2}{2}\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}\begin{pmatrix}U\\Q\\V\\S\end{pmatrix},\tag{6.1}$$

where $\kappa_i = \mathrm{d}k_i/\mathrm{d}t$ is a constant, and the γ_i are the constants of proportionality between the w_{ij}. In Case I, $w(\xi_1) = 1$, while in Case II, $\gamma_1 = -\gamma_4$ and $\gamma_1^2+\gamma_2^2+\gamma_3^2 = 0$.

Set

$$\mathsf{K} = \begin{pmatrix}0&\gamma_1&\gamma_2&\gamma_3\\-\gamma_1&0&-\gamma_3&\gamma_2\\-\gamma_2&\gamma_3&0&\gamma_4\\-\gamma_3&-\gamma_2&-\gamma_4&0\end{pmatrix},\qquad \mathsf{J}_0 = \begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}.$$

In all cases $[\mathsf{K},\mathsf{J}_0] = 0$. In Case II, $\gamma_1 = -\gamma_4$, and defining J_i for $i = 1, 2, 3$ by $\mathsf{K}=\gamma_1\mathsf{J}_1+\gamma_2\mathsf{J}_2+\gamma_3\mathsf{J}_3$ yields

$$[\mathsf{J}_1,\mathsf{J}_2] = 2\mathsf{J}_3,\qquad [\mathsf{J}_2,\mathsf{J}_3] = 2\mathsf{J}_1,\qquad [\mathsf{J}_3,\mathsf{J}_1] = 2\mathsf{J}_2,$$

and thus K is an element of a 4×4 representation of $\mathfrak{sl}(2,\mathbb{C})$. The condition $\gamma_1^2+\gamma_2^2+\gamma_3^2 = 0$ implies that $\mathsf{K}^2 = 0$.

The right-hand side of (6.1) can be written in terms of $\mathfrak{sl}(2,\mathbb{C})$ and J_0. In particular the right-hand side of (6.1) is given by

$$\frac{w(\xi_1)}{k_1y-x+k_2}\begin{pmatrix}0&\frac{1}{2}(\gamma_1-\gamma_4)&\gamma_2&\gamma_3\\-\frac{1}{2}(\gamma_1-\gamma_4)&0&-\gamma_3&\gamma_2\\-\gamma_2&\gamma_3&0&-\frac{1}{2}(\gamma_1-\gamma_4)\\-\gamma_3&-\gamma_2&\frac{1}{2}(\gamma_1-\gamma_4)&0\end{pmatrix}$$
$$+\frac{1}{2}\left(\frac{w(\xi_1)}{k_1y-x+k_2}(\gamma_1+\gamma_4)+\kappa_4-\xi_1\kappa_2\right)\mathsf{J}_0 = \alpha'\mathsf{A}+\beta'\mathsf{J}_0,$$

which defines $\mathsf{A}\in\mathfrak{sl}(2,\mathbb{C})$ (in Case II, $\mathsf{A} = \mathsf{K}$), where

$$\alpha' = \frac{w(\xi_1)}{k_1y-x+k_2},\qquad \beta' = \begin{cases}\frac{1}{2}\left(\dfrac{\gamma_1+\gamma_4}{k_1y-x+k_2}+\kappa_4-\xi_1\kappa_2\right) & \text{Case I,}\\ \frac{1}{2}(\kappa_4-\xi_1\kappa_2) & \text{Case II.}\end{cases}\tag{6.2}$$

We have deliberately used primed α and β variables as they are shortly to be integrated. Thus the right-hand side of (6.1) is of the form of a sum of an element A in a 4×4 representation of $\mathfrak{sl}(2,\mathbb{C})$ and J_0. In addition $\mathsf{A}^2 = \kappa^2\mathsf{I}_4$ where $\kappa^2 = -\left[\frac{1}{4}(\gamma_1-\gamma_4)^2+\gamma_2^2+\gamma_3^2\right]$ and $[\mathsf{A},\mathsf{J}_0] = 0$.

The system is solved by the method of characteristics. It does not decouple, but is linear. Let $\boldsymbol{U} = (U,Q,V,S)^{\mathrm{T}}$ and set $X = x-k_2$, $Y = y$, $Z = z+k_4$. This amounts to using classical symmetries to set k_2 and k_4 to zero. The characteristic system of equations (6.1) is the system of ODEs,

$$\frac{\mathrm{d}X}{\mathrm{d}\lambda} = Z+k_3Y,\qquad \frac{\mathrm{d}Y}{\mathrm{d}\lambda} = -k_1Z-k_3X,\qquad \frac{\mathrm{d}Z}{\mathrm{d}\lambda} = k_1Y-X,$$
$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = 0,\qquad \frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}\lambda} = (k_1Y-X)\left(\alpha'\mathsf{A}+\beta'\mathsf{J}_0\right)\boldsymbol{U}.\tag{6.3}$$

Invariants of the first four equations are

$$t,\qquad \chi_1 = k_1X+Y-k_3Z,\qquad \chi_2 = X^2+Y^2+Z^2.\tag{6.4}$$

The third and fifth equations can be combined to yield

$$\frac{dU}{dZ} = \alpha' \mathsf{A} + \beta' \mathsf{J}_0 .$$

In each of the cases, one writes α' and β' as a function of (Z, χ_1, χ_2, t), and integrates them with respect to Z, with the χ_i and t regarded as constants, to obtain

$$U = \exp(\alpha \mathsf{A}) \exp(\beta \mathsf{J}_0) \boldsymbol{u} , \tag{6.5}$$

where

$$\frac{d}{dZ}\alpha = \alpha' , \qquad \frac{d}{dZ}\beta = \beta' . \tag{6.6}$$

We now calculate α and β in Case I and the four subcases of Case II. The integration of α' leads naturally to four subcases of Case I, detailed below.

Case Ia: $w(\xi_1) = 1$, $\mu = 1 + k_1^2 + k_3^2 \neq 0$, $1 + k_1^2 \neq 0$. In this case, from (6.4) we have $(k_1 Y - X)^2 = (1 + k_1^2)(\chi_2 - \chi_1^2/\mu) - \mu(Z + k_3\chi_1/\mu)^2$. Eliminating X using $k_1 X + Y = \chi_1 + k_3 Z$ yields an expression for Y in terms of Z, χ_1 and χ_2. Hence we obtain the identity

$$\frac{k_3 Y + Z}{k_1 Y - X} = \frac{k_1 k_3}{1 + k_1^2} + \frac{k_3 \chi_1 + \mu Z}{(1 + k_1^2)\sqrt{(1 + k_1^2)(\chi_2 - \chi_1^2/\mu) - \mu(Z + k_3\chi_1/\mu)^2}} .$$

From this expression we can integrate α' and β' with respect to Z regarding χ_1 and χ_2 as constants as required. Thus

$$\alpha = \frac{1}{\sqrt{\mu}} \arcsin\left(\frac{\mu Z + k_3 \chi_1}{\sqrt{1 + k_1^2}\sqrt{\mu\chi_2 - \chi_1^2}} \right) ,$$

$$\beta = \frac{(\gamma_1 + \gamma_4)}{2}\alpha + \frac{\kappa_4 Z}{2} - \frac{\kappa_2 X}{2} + \frac{\kappa_2 k_1}{2(1 + k_1^2)}\chi_1 .$$

Case Ib: $w(\xi_1) = 1$, $\mu = 1 + k_1^2 + k_3^2 = 0$, $1 + k_1^2 \neq 0$. In this case, we have $(k_1 Y - X)^2 = -\chi_1^2 - k_3^2\chi_2 - 2\chi_1 k_3 Z$ and

$$k_3 Y + Z = \frac{k_3 \chi_1}{1 + k_1^2} + \frac{k_1 k_3}{1 + k_1^2}(k_1 Y - X) .$$

Using these expressions gives

$$\alpha = -\frac{k_1 Y - X}{k_3 \chi_1} , \qquad \beta = \tfrac{1}{2}(\gamma_1 + \gamma_4)\alpha + \frac{\kappa_2}{2(1 + k_1^2)}(k_1 Y - X) + \frac{\kappa_4 k_3 + \kappa_2 k_1}{2k_3} Z .$$

Case Ic: $w(\xi_1) = 1$, $k_3 \neq 0$, $1 + k_1^2 = 0$. In this case, we have $k_1 Y - X = k_1(k_1 X + Y) = k_1(\chi_1 + k_3 Z)$. Eliminating X^2 from

$$X^2 + Y^2 = \chi_2 - Z^2 , \qquad (k_1 Y - X)^2 = -(\chi_1 + k_3 Z)^2 ,$$

gives

$$Y = \frac{1}{2}\frac{\chi_2 - Z^2}{\chi_1 + k_3 Z} + \tfrac{1}{2}(\chi_1 + k_3 Z) .$$

Thus $\xi_1 = (k_3 Y + Z)/(k_1 Y - X)$ can be obtained in terms of Z, χ_1 and χ_2, to yield

$$\alpha = \frac{\log(\chi_1 + k_3 Z)}{k_1 k_3},$$

$$\beta = \tfrac{1}{2}(\gamma_1+\gamma_4)\alpha + \tfrac{1}{2}\kappa_4 Z - \frac{\kappa_2}{4k_1k_3}(1+k_3^2)Z - \frac{\chi_1^2 - k_3^2\chi_2}{4k_1k_3^2(\chi_1+k_3Z)}.$$

Case Id: $w(\xi_1) = 1$, $k_3 = 0$, $1 + k_1^2 = 0$. In this case we have $k_1Y - X = k_1\chi_1$ and $\xi_1 = Z/(k_1Y - X) = Z/(k_1\chi_1)$. Hence

$$\alpha = \frac{-k_1 Z}{\chi_1}, \qquad \beta = \tfrac{1}{2}(\gamma_1+\gamma_4)\alpha + \tfrac{1}{2}\kappa_4 Z + \frac{\kappa_2 k_1 Z^2}{4\chi_1}.$$

We now turn our attention to Case II. Recall Case II has $\gamma_1 + \gamma_4 = 0$, and $A^2 = 0$.

Case IIa: $\mu = 1 + k_1^2 + k_3^2 \neq 0$, $1 + k_1^2 \neq 0$. We write $k_1Y - X$ and ξ_1 in terms of Z, the χ_i and t as for Case Ia. The function w in the expression (6.2) for α' is of the form (5.4). Hence

$$\alpha = \frac{1}{2\mu}\left[\arcsin\left(\frac{\mu Z + k_3\chi_1}{\sqrt{1+k_1^2}\sqrt{\mu\chi_2 - \chi_1^2}}\right)\right]^2 = \frac{1}{2\mu}\left[\arctan\left(\frac{\mu Z + k_3\chi_1}{\sqrt{\mu}(k_1Y - X)}\right)\right]^2,$$

$$\beta = \frac{\kappa_4 Z}{2} - \frac{\kappa_2 X}{2} + \frac{\kappa_2 k_1}{2(1+k_1^2)}\chi_1.$$

Case IIb: $\mu = 1 + k_1^2 + k_3^2 = 0$, $k_3 \neq 0$. We write $k_1Y - X$ and ξ_1 in terms of Z, the χ_i and t as for Case Ib. The function w in the expression (6.2) for α' is of the form (5.5). Hence

$$\alpha = \frac{Z}{k_3\chi_1}, \qquad \beta = \frac{\kappa_4 Z}{2} - \frac{\kappa_2}{2k_3^2}(k_1\chi_1 + k_3^2 X).$$

Case IIc: $k_3 \neq 0$, $1 + k_1^2 = 0$. We write $k_1Y - X$ and ξ_1 in terms of Z, the χ_i and t as for Case Ic. The function w in the expression (6.2) for α' is of the form (5.6). Hence

$$\alpha = -\frac{1}{4k_1k_3}\left[\log\left(\frac{\chi_1^2 - k_3^2\chi_2}{(\chi_1 + k_3Z)^2}\right)\right]^2,$$

$$\beta = \frac{\kappa_4 Z}{2} - \frac{\kappa_2(1+k_3^2)Z}{4k_1k_3} - \frac{\kappa_2}{4}\,\frac{\chi_1^2 - k_3^2\chi_2}{k_3^2 k_1(\chi_1 + k_3 Z)}.$$

Case IId: $k_3 = 0$, $1 + k_1^2 = 0$. We write $k_1Y - X$ and ξ_1 in terms of Z, the χ_i and t as for Case Id. The function w in the expression (6.2) for α' is of the form (5.7). Then

$$\alpha = -\frac{Z^2}{2\chi_1^2}, \qquad \beta = \frac{\kappa_4 Z}{2} - \frac{\kappa_2 Z^2}{4k_1\chi_1}.$$

6.2. Singular case $\xi_{1,x} = 0$, $1 + \xi_y^2 \neq 0$, $\xi_{2,x} \neq 0$

Using the infinitesimals obtained in Section 5.2, the ISCs can be written as

$$\begin{pmatrix} U_x & U_y & U_z & U_t \\ Q_x & Q_y & Q_z & Q_t \\ V_x & V_y & V_z & V_t \\ S_x & S_y & S_z & S_t \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ 1 \\ 0 \end{pmatrix} = \frac{w(\xi_2)}{g(t) - y}\begin{pmatrix} 0 & \gamma_1 & \gamma_2 & \gamma_3 \\ -\gamma_1 & 0 & -\gamma_3 & \gamma_2 \\ -\gamma_2 & \gamma_3 & 0 & \gamma_4 \\ -\gamma_3 & -\gamma_2 & -\gamma_4 & 0 \end{pmatrix}\begin{pmatrix} U \\ Q \\ V \\ S \end{pmatrix}$$

$$+\frac{\kappa_4-\xi_2\kappa_2}{2}\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}\begin{pmatrix}U\\Q\\V\\S\end{pmatrix}. \tag{6.7}$$

Here ξ_1 is a constant, ξ_2 is given by (5.8), $\kappa_2 = \mathrm{d}h/\mathrm{d}t$ and $\kappa_4 = \mathrm{d}g/\mathrm{d}t$ are constants, and the γ_i are the constants of proportionality between the w_{ij}. In Case III, $w(\xi_2) = 1$ while in Case IV, $\gamma_1 = -\gamma_4$ and $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 0$.

Following the argument in the previous subsection, setting

$$X = x, \qquad Y = \kappa_4 t + \kappa_6 - y, \qquad Z = z + \kappa_2 t + \kappa_5, \tag{6.8}$$

where the κ_i are constants, the invariants of (6.7) are $\chi_1 = X/\xi_1 - Z$ and $\chi_2 = (\xi_1 X + Z)^2 + (1+\xi_1^2)Y^2$. We note again in this singular case, ξ_1 is a constant. Then the solution of (6.7) giving the reduction in this case is (6.5), (6.6), where now

$$\alpha' = \frac{w(\xi_2)}{Y}, \qquad \beta' = \tfrac{1}{2}\big((\gamma_1+\gamma_4)\alpha' + \kappa_2 - \kappa_4\xi_2\big).$$

As before, we need to write α' and β' in terms of Z and the χ_i, and then integrate with respect to Z to obtain the final form of (6.5). From (6.8) we have

$$Y = \frac{\sqrt{\chi_2 - (\xi_1^2\chi_1 + (1+\xi_1^2)Z)^2}}{\sqrt{1+\xi_1^2}}.$$

Further, we can write

$$\xi_2 = \frac{\sqrt{1+\xi_1^2}\,(\xi_1^2\chi_1 + (1+\xi_1^2)Z)}{\sqrt{\chi_2 - (\xi_1^2\chi_1 + (1+\xi_1^2)Z)^2}}.$$

Case III: Since $w(\xi_2) = 1$,

$$\alpha = \frac{1}{\sqrt{1+\xi_1^2}}\arcsin\left(\frac{\xi_1 X + Z}{\sqrt{\chi_2}}\right), \qquad \beta = \tfrac{1}{2}(\gamma_1+\gamma_4)\alpha + \tfrac{1}{2}\kappa_2 Z - \tfrac{1}{2}\kappa_4 Y.$$

Case IV: In this case $\gamma_1 + \gamma_4 = 0$ and $w(\xi_2)$ is given by (5.9). We obtain

$$\alpha = \frac{1}{2}\left(\arctan\left(\frac{\xi_1 X + Z}{\sqrt{\chi_2}}\right)\right)^2, \qquad \beta = \frac{\kappa_2 Z}{2} - \frac{\kappa_4 Y}{2}.$$

7. The reduced equations, Case 2

Let $\boldsymbol{U} = (U_0, Q_0, V_0, S_0)^{\mathrm{T}}$ where U_0, Q_0, V_0, S_0 are functions of χ_1, χ_2, t and α and β are as calculated in the previous section. Substituting

$$\boldsymbol{U} = \exp(\alpha\mathsf{A})\exp(\beta\mathsf{J}_0)\boldsymbol{u}$$

into (2.1), we obtain the sought-for reduction equation with $\boldsymbol{u}$ as the dependent variable.

Since $[\mathsf{A}, \mathsf{J}_0] = 0$, the exponentials of A and J_0 commute.

In Cases I and III, setting $\gamma_{14} = \frac{1}{2}(\gamma_1 - \gamma_4)$ gives

$$\mathsf{A} = \begin{pmatrix} 0 & \gamma_{14} & \gamma_2 & \gamma_3 \\ -\gamma_{14} & 0 & -\gamma_3 & \gamma_2 \\ -\gamma_2 & \gamma_3 & 0 & -\gamma_{14} \\ -\gamma_3 & -\gamma_2 & \gamma_{14} & 0 \end{pmatrix}$$

and with $\kappa = \sqrt{-\gamma_{14}^2 - \gamma_2^2 - \gamma_3^2}$ we have $\exp(\alpha\mathsf{A}) = \kappa^{-1}\sinh(\kappa\alpha)\mathsf{A} + \cosh(\kappa\alpha)\mathsf{I}_4$. In Cases II and IV, $\mathsf{A}^2 = 0$ implies that

$$\exp(\alpha\mathsf{A}) = I + \alpha\mathsf{A}\,.$$

Further, $\exp(\beta\mathsf{J}_0)$ is a rotation matrix

$$\exp(\beta\mathsf{J}_0) = \begin{pmatrix} \cos(\beta) & \sin(\beta) & 0 & 0 \\ -\sin(\beta) & \cos(\beta) & 0 & 0 \\ 0 & 0 & \cos(\beta) & \sin(\beta) \\ 0 & 0 & -\sin(\beta) & \cos(\beta) \end{pmatrix}.$$

In matrix notation, the system (2.1) becomes

$$-\mathsf{J}_0\boldsymbol{U}_t + \nabla^2\boldsymbol{U} + \mathcal{R}^2\boldsymbol{U} = 0\,.$$

Set $\boldsymbol{U} = \mathsf{M}\boldsymbol{u}$ and $\mathsf{M} = \exp(\alpha\mathsf{A} + \beta\mathsf{J}_0)$. Note that

$$-\frac{\partial}{\partial t}\mathsf{J}_0\mathsf{M}\boldsymbol{u} = -(\alpha_t\mathsf{A}\mathsf{J}_0 - \beta_t\mathsf{I}_4)\mathsf{M}\boldsymbol{u} - \mathsf{J}_0\mathsf{M}\left(\chi_{1,t}\boldsymbol{u}_{\chi_1} + \chi_{2,t}\boldsymbol{u}_{\chi_2} + \boldsymbol{u}_t\right).$$

To calculate $\nabla^2\boldsymbol{U}$, we note that

$$\begin{aligned}\nabla^2\mathsf{M}\boldsymbol{u} = {}& \left(\nabla^2\alpha\mathsf{A} + \nabla^2\beta\mathsf{J}_0\right)\mathsf{M}\boldsymbol{u} + \left(\left(\kappa^2\nabla\alpha\cdot\nabla\alpha - \nabla\beta\cdot\nabla\beta\right)\mathsf{I}_4 + 2\nabla\alpha\cdot\nabla\beta\mathsf{A}\mathsf{J}_0\right)\mathsf{M}\boldsymbol{u} \\ & + 2\mathsf{M}\left(\nabla\alpha\cdot\nabla\chi_1\mathsf{A}\boldsymbol{u}_{\chi_1} + \nabla\alpha\cdot\nabla\chi_2\mathsf{A}\boldsymbol{u}_{\chi_2} + \nabla\beta\cdot\nabla\chi_1\mathsf{J}_0\boldsymbol{u}_{\chi_1} + \nabla\beta\cdot\nabla\chi_2\mathsf{J}_0\boldsymbol{u}_{\chi_2}\right) \\ & + \mathsf{M}\left(\nabla\chi_1\cdot\nabla\chi_1\boldsymbol{u}_{\chi_1\chi_1} + 2\nabla\chi_1\cdot\nabla\chi_2\boldsymbol{u}_{\chi_1\chi_2} + \nabla\chi_2\cdot\nabla\chi_2\boldsymbol{u}_{\chi_2\chi_2} + \nabla^2\chi_1\boldsymbol{u}_{\chi_1} + \nabla^2\chi_2\boldsymbol{u}_{\chi_2}\right).\end{aligned}$$

A simple calculation yields $\mathcal{R}^2 \equiv U_0^2 + Q_0^2 + V_0^2 + S_0^2$ in all cases. Thus $\mathcal{R}^2$ is invariant and we can replace $\mathcal{R}^2$ in (1.1) by any function of $\mathcal{R}^2$ and still obtain a reduction. In particular, we may have a saturation term $\mathcal{R}^2/(1+\mathcal{R}^2)$ instead of a cubic nonlinearity in (1.1).

Setting

$$\begin{pmatrix} U_0 + \mathrm{i}Q_0 \\ V_0 + \mathrm{i}S_0 \end{pmatrix} = \boldsymbol{\varphi}\,, \qquad \widehat{\mathsf{A}} = \begin{pmatrix} \gamma_1 & \gamma_2 + \mathrm{i}\gamma_3 \\ \gamma_2 - \mathrm{i}\gamma_3 & -\gamma_1 \end{pmatrix},$$

say, we can use $\mathsf{J}_0{}^2 = -\mathsf{I}_4$ to write the 1×4 and 4×4 terms in the reduction in 1×2 and 2×2 form as

$$\mathsf{J}_0\boldsymbol{u} \leftrightarrow \mathrm{i}\boldsymbol{\varphi}\,, \qquad \mathsf{A}\mathsf{J}_0\boldsymbol{u} \leftrightarrow \widehat{\mathsf{A}}\boldsymbol{\varphi}\,, \qquad \mathsf{A}\boldsymbol{u} \leftrightarrow -\mathrm{i}\widehat{\mathsf{A}}\boldsymbol{\varphi}\,. \tag{7.1}$$

7.1. Generic case $\xi_{1,x} \neq 0$

Recall the new independent variables are $\chi_1 = k_1X + Y - k_3Z$ and $\chi_2 = X^2 + Y^2 + Z^2$ where $X = x + k_2(t)$, $Y = y$, $Z = z + k_4(t)$ and k_1, k_3 are constants. In Case I, $A^2 = \kappa^2 I_4$ where $\kappa^2 = -\left[\frac{1}{4}(\gamma_1 - \gamma_4)^2 + \gamma_2^2 + \gamma_3^2\right]$. In Case II, $\mathsf{A}^2 = 0$, that is $\kappa^2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 0$ and $\gamma_1 = -\gamma_4$.

The template for the reduction is

$$
\begin{aligned}
0 = &-\mathrm{J}_0 \boldsymbol{u}_t + \mu \boldsymbol{u}_{\chi_1\chi_1} + 4\chi_1 \boldsymbol{u}_{\chi_1\chi_2} + 4\chi_2 \boldsymbol{u}_{\chi_2\chi_2} \\
&+ \left(2\nabla\alpha\cdot\nabla\chi_1 \mathrm{A} + (2\nabla\beta\cdot\nabla\chi_1 - \chi_{1,t})\mathrm{J}_0\right)\boldsymbol{u}_{\chi_1} + \left(2\nabla\alpha\cdot\nabla\chi_2 \mathrm{A} + (2\nabla\beta\cdot\nabla\chi_2 - \chi_{2,t})\mathrm{J}_0 + 6\right)\boldsymbol{u}_{\chi_2} \\
&+ \left(\nabla^2\alpha \mathrm{A} + \nabla^2\beta \mathrm{J}_0 - (\alpha_t - 2\nabla\alpha\cdot\nabla\beta)\mathrm{A}\mathrm{J}_0 + \kappa^2\nabla\alpha\cdot\nabla\alpha + \beta_t - \nabla\beta\cdot\nabla\beta + \mathcal{R}^2\right)\boldsymbol{u},
\end{aligned}
$$

where we have used $\mu = 1 + k_1^2 + k_3^2$ and

$$
\begin{aligned}
&\nabla^2\chi_1 = 0, \qquad \nabla^2\chi_2 = 6, \qquad \nabla\chi_1\cdot\nabla\chi_1 = \mu, \\
&\nabla\chi_1\cdot\nabla\chi_2 = 2\chi_1, \qquad \nabla\chi_2\cdot\nabla\chi_2 = 4\chi_2.
\end{aligned}
$$

Since $k_2 = \kappa_2 t + \kappa_5$, $k_4 = \kappa_4 t + \kappa_6$ where the κ_i are constants, we have

$$
\chi_{1,t} = -k_1\kappa_2 - k_3\kappa_4, \qquad \chi_{2,t} = -2\kappa_2 X + 2\kappa_4 Z.
$$

Case Ia: $\mu = 1 + k_1^2 + k_3^2 \neq 0$, $1 + k_1^2 \neq 0$. Using the form for α and β obtained in the previous section, but now regarding the χ_i as functions of X, Y, Z and t, the reduced form of the CNLS system is

$$
\begin{aligned}
0 = &-\mathrm{J}_0 \boldsymbol{u}_t + \mu \boldsymbol{u}_{\chi_1\chi_1} + 4\chi_1 \boldsymbol{u}_{\chi_1\chi_2} + 4\chi_2 \boldsymbol{u}_{\chi_2\chi_2} + \frac{\kappa_2 k_1 \mu}{1 + k_1^2}\mathrm{J}_0 \boldsymbol{u}_{\chi_1} + \frac{2\kappa_2 k_1}{1 + k_1^2}\chi_1 \mathrm{J}_0 \boldsymbol{u}_{\chi_2} + 6\boldsymbol{u}_{\chi_2} \\
&+ \left(-\frac{(\gamma_1 + \gamma_4)}{2(\mu\chi_2 - \chi_1^2)}\mathrm{A}\mathrm{J}_0 + \frac{4\kappa_2^2 - (\gamma_1 + \gamma_4)^2}{4\mu\chi_2 - \chi_1^2} + \frac{\kappa_2^2 + \kappa_4^2}{4} - \frac{\mu\kappa_2 k_1^2}{4(1 + k_1^2)} + \mathcal{R}^2\right)\boldsymbol{u}.
\end{aligned}
$$

Making the change of coordinates $\rho = \mu\chi_2 - \chi_1^2$, $\sigma = \chi_1$, the reduction becomes

$$
\begin{aligned}
0 = &-\mathrm{J}_0 \boldsymbol{u}_t + \mu(\boldsymbol{u}_{\sigma\sigma} - 4\boldsymbol{u}_\rho) + \frac{\kappa_2 k_1 \mu}{1 + k_1^2}\mathrm{J}_0 \boldsymbol{u}_\sigma \\
&+ \left(-\frac{(\gamma_1 + \gamma_4)}{2\rho}\mathrm{A}\mathrm{J}_0 + \frac{\kappa_2^2 - \frac{1}{4}(\gamma_1 + \gamma_4)^2}{\rho} + \frac{\kappa_2^2 + \kappa_4^2}{4} - \frac{\mu\kappa_2 k_1^2}{4(1 + k_1^2)} + \mathcal{R}^2\right)\boldsymbol{u}.
\end{aligned}
$$

Using Table (7.1), the reduction can be written in the 2×2 form,

$$
\begin{aligned}
0 = &\,\mathrm{i}(\varphi_t + \delta_1\varphi_\sigma + \delta_2\varphi_\rho) \\
&+ \delta_2\varphi_{\sigma\sigma} - \frac{\delta_3}{\rho}\begin{pmatrix} \gamma_1 & (\gamma_2 + \mathrm{i}\gamma_3) \\ (\gamma_2 - \mathrm{i}\gamma_3) & -\gamma_1 \end{pmatrix}\varphi + \frac{\delta_4}{\rho}\varphi + \delta_5\varphi + |\varphi|^2\varphi,
\end{aligned}
$$

where the δ_i are constants. This is almost the 1+1-dimensional nonlinear Schrödinger equation, except for two terms, both with a singularity at $\rho = 0$, and one with a coupling matrix in $\mathfrak{sl}(2, \mathbb{C})$.

It is interesting to note that, in the notation of (6.3),

$$
\rho = \mu\chi_2 - \chi_1^2 = \left(\frac{\mathrm{d}X}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}Y}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}Z}{\mathrm{d}\lambda}\right)^2,
$$

which is the infinitesimal arc length along the solutions of the characteristics of the ISC.

Case Ib: $\mu = 1 + k_1^2 + k_3^2 = 0$, $1 + k_1^2 \neq 0$. The reduced CNLS system is

$$
\begin{aligned}
0 = &-\mathrm{J}_0 \boldsymbol{u}_t + 4\chi_1 \boldsymbol{u}_{\chi_1\chi_2} + 4\chi_2 \boldsymbol{u}_{\chi_2\chi_2} + \frac{2k_1\kappa_2\chi_1}{1 + k_1^2}\mathrm{J}_0 \boldsymbol{u}_{\chi_2} + 6\boldsymbol{u}_{\chi_2} \\
&+ \left(-\frac{\gamma_1 + \gamma_4}{\chi_1^2}\mathrm{A}\mathrm{J}_0 + \frac{\kappa_4^2 + \kappa_2^2}{4} + \frac{(\gamma_1 + \gamma_4)^2}{4\chi_1^2} - \frac{\kappa^2}{\chi_1^2} + \mathcal{R}^2\right)\boldsymbol{u}.
\end{aligned}
$$

Case Ic: $k_3 \neq 0$, $1 + k_1^2 = 0$ (so $\mu^2 = k_3^2$). The reduced CNLS system is

$$0 = -\mathsf{J}_0 \boldsymbol{u}_t + k_3^2 \boldsymbol{u}_{\chi_1\chi_1} + 4\chi_1 \boldsymbol{u}_{\chi_1\chi_2} + 4\chi_2 \boldsymbol{u}_{\chi_2\chi_2} + \frac{\kappa_2(k_3^2-1)}{2k_1} \mathsf{J}_0 \boldsymbol{u}_{\chi_1} + \left(\frac{4}{k_1 k_3} \mathsf{A} + 6 + \left(2\frac{\gamma_1+\gamma_4}{k_1 k_3} + \kappa_2 \chi_1 \frac{k_3^2-1}{k_3^2 k_1} \right) \mathsf{J}_0 \right) \boldsymbol{u}_{\chi_2} + \left(\frac{\kappa_4^2}{4} + \frac{\kappa_2^2 (1+k_3^2)^2}{16 k_3^2} + \mathcal{R}^2 \right) \boldsymbol{u}.$$

Case Id: $k_3 = 0$, $1 + k_1^2 = 0$. The reduced CNLS system is

$$0 = -\mathsf{J}_0 \boldsymbol{u}_t + 4\chi_1 \boldsymbol{u}_{\chi_1\chi_2} + 4\chi_2 \boldsymbol{u}_{\chi_2\chi_2} + \frac{k_1}{\kappa_2} \mathsf{J}_0 \boldsymbol{u}_{\chi_1} + \frac{\kappa_2(\chi_1^2 - \chi_2)}{k_1 \chi_1} \mathsf{J}_0 \boldsymbol{u}_{\chi_2} + 6\boldsymbol{u}_{\chi_2} + \left(\frac{k_1 \kappa_2}{2\chi_1} \mathsf{J}_0 - \frac{\gamma_1+\gamma_4}{\chi_1^2} \mathsf{A}\mathsf{J}_0 - \frac{\kappa^2}{\chi_1^2} + \frac{\kappa_4^2}{4} + \frac{(\gamma_1+\gamma_4)^2}{4\chi_1^2} + \mathcal{R}^2 \right) \boldsymbol{u}.$$

We now list the reductions for Case II. Recall that $\mathsf{A}^2 = 0$ in this case.

Case IIa: $\mu = 1 + k_1^2 + k_3^2 \neq 0$, $1 + k_1^2 \neq 0$. The reduced CNLS system is

$$0 = -\mathsf{J}_0 \boldsymbol{u}_t + \mu \boldsymbol{u}_{\chi_1\chi_1} + 4\chi_1 \boldsymbol{u}_{\chi_1\chi_2} + 4\chi_2 \boldsymbol{u}_{\chi_2\chi_2} + \frac{k_1 k_3 \mu}{1+k_1^2} \mathsf{J}_0 \boldsymbol{u}_{\chi_1} + \frac{2 k_1 k_3 \chi_1}{1+k_1^2} \mathsf{J}_0 \boldsymbol{u}_{\chi_2} + 6\boldsymbol{u}_{\chi_2} + \left(\frac{2\mu}{\mu\chi_2 - \chi_1^2} \mathsf{A} + \frac{\kappa_2^2 + \kappa_4^2}{4} - \frac{\mu k_1^2 k_3^2}{4(1+k_1^2)} + \mathcal{R}^2 \right) \boldsymbol{u}.$$

Case IIb: $\mu = 1 + k_1^2 + k_3^2 = 0$, $k_3 \neq 0$. The reduced CNLS system is

$$0 = -\mathsf{J}_0 \boldsymbol{u}_t + 4\chi_1 \boldsymbol{u}_{\chi_1\chi_2} + 4\chi_2 \boldsymbol{u}_{\chi_2\chi_2} - \frac{2}{\chi_1} \mathsf{A} \boldsymbol{u}_{\chi_1} - \frac{2 k_1 \kappa_2 \chi_1}{k_3^2} \mathsf{J}_0 \boldsymbol{u}_{\chi_2} + 6\boldsymbol{u}_{\chi_2} + \left(\frac{2}{\chi_1^2} \mathsf{A} + \left(\frac{\kappa_2 k_1}{k_3^2 \chi_1} \right) \mathsf{A}\mathsf{J}_0 + \frac{\kappa_2^2 + \kappa_4^2}{4} + \mathcal{R}^2 \right) \boldsymbol{u}.$$

Case IIc: $k_3 \neq 0$, $1 + k_1^2 = 0$ (so $\mu^2 = k_3^2$). The reduced CNLS system is

$$0 = -\mathsf{J}_0 \boldsymbol{u}_t + k_3^2 \boldsymbol{u}_{\chi_1\chi_1} + 4\chi_1 \boldsymbol{u}_{\chi_1\chi_2} + 4\chi_2 \boldsymbol{u}_{\chi_2\chi_2} + \tfrac{1}{2} k_1 \kappa_2 (1 - k_3^2) \mathsf{J}_0 \boldsymbol{u}_{\chi_1} + 6\boldsymbol{u}_{\chi_2} + \frac{\kappa_2 \chi_1 (k_3^2 - 1)}{k_3^2 k_1} \mathsf{J}_0 \boldsymbol{u}_{\chi_2} + \left(\frac{2 k_1 k_3}{\chi_1^2 - k_3^2 \chi_2} \mathsf{A} + \frac{\kappa_4^2}{4} + \kappa_2^2 \frac{(1+k_3^2)^2}{16 k_3^2} + \mathcal{R}^2 \right) \boldsymbol{u}.$$

Case IId: $k_3 = 0$, $1 + k_1^2 = 0$. The reduced CNLS system is

$$0 = -\mathsf{J}_0 \boldsymbol{u}_t + 4\chi_1 \boldsymbol{u}_{\chi_1\chi_2} + 4\chi_2 \boldsymbol{u}_{\chi_2\chi_2} + (k_3 \kappa_4 + k_1 \kappa_2) \mathsf{J}_0 \boldsymbol{u}_{\chi_1} + 6\boldsymbol{u}_{\chi_2} + \kappa_2 \frac{\chi_1^2 - \chi_2}{k_1 \chi_1} \mathsf{J}_0 \boldsymbol{u}_{\chi_2} + \left(-\frac{1}{\chi_1^2} \mathsf{A} + \frac{k_1 \kappa_2}{2\chi_1} \mathsf{J}_0 + \frac{\kappa_4^2}{4} + \mathcal{R}^2 \right) \boldsymbol{u}.$$

7.2. *Singular case $\xi_{1,x} = 0$, $1 + \xi_1^2 \neq 0$, $\xi_{2,x} \neq 0$*

Recall the new independent variables are $\chi_1 = X/\xi_1 - Z$ and $\chi_2 = (\xi_1 X + Z)^2 + (1+\xi_1^2) Y^2$ where $X = x$, $Y = \kappa_4 t + \kappa_6 - y$, $Z = z + \kappa_2 t + \kappa_5$ and ξ_1 is a constant. In Case III, $\mathsf{A}^2 = \kappa^2 \mathsf{I}_4$ where $\kappa^2 = -\left[\frac{1}{4}(\gamma_1 - \gamma_4)^2 + \gamma_2^2 + \gamma_3^2\right]$ In Case IV, $\mathsf{A}^2 = 0$, that is, $\kappa^2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 0$ and $\gamma_1 = -\gamma_4$.

The template for the reduction is

$$0 = -\mathrm{J}_0 \boldsymbol{u}_t + \frac{1+\xi_1^2}{\xi_1^2} \boldsymbol{u}_{\chi_1\chi_1} + 4(1+\xi_1^2)\chi_2 \boldsymbol{u}_{\chi_2\chi_2} + \left(2\nabla\alpha\cdot\nabla\chi_1 \mathrm{A} + (2\nabla\beta\cdot\nabla\chi_1 - \chi_{1,t})\mathrm{J}_0\right)\boldsymbol{u}_{\chi_1}$$
$$+ \left(2\nabla\alpha\cdot\nabla\chi_2 \mathrm{A} + (2\nabla\beta\cdot\nabla\chi_2 - \chi_{2,t})\mathrm{J}_0 + 4(1+\xi_1^2)\right)\boldsymbol{u}_{\chi_2}$$
$$+ \left(\nabla^2\alpha \mathrm{A} + \nabla^2\beta \mathrm{J}_0 - (\alpha_t - 2\nabla\alpha\cdot\nabla\beta)\mathrm{A}\mathrm{J}_0 + \kappa^2\nabla\alpha\cdot\nabla\alpha + \beta_t - \nabla\beta\cdot\nabla\beta + \mathcal{R}^2\right)\boldsymbol{u}.$$

Case III: $\kappa^2 \neq 0$. The reduced CNLS system is

$$0 = -\mathrm{J}_0 \boldsymbol{u}_t + \frac{1+\xi_1^2}{\xi_1^2} \boldsymbol{u}_{\chi_1\chi_1} + 4(1+\xi_1^2)\chi_2 \boldsymbol{u}_{\chi_2\chi_2} + 4(1+\xi_1^2)\boldsymbol{u}_{\chi_2}$$
$$+ \left(\frac{\gamma_1+\gamma_4}{\chi_2}\mathrm{A}\mathrm{J}_0 + \frac{\kappa^2}{\chi_2} - \frac{(\gamma_1+\gamma_4)^2}{4\chi_2} + \frac{\kappa_2^2+\kappa_4^2}{4} + \mathcal{R}^2\right)\boldsymbol{u}.$$

Case IV: $\kappa^2 = 0$, $\gamma_1 + \gamma_4 = 0$. The reduced CNLS system is

$$0 = -\mathrm{J}_0 \boldsymbol{u}_t + \frac{1+\xi_1^2}{\xi_1^2} \boldsymbol{u}_{\chi_1\chi_1} + 4(1+\xi_1^2)\chi_2 \boldsymbol{u}_{\chi_2\chi_2} + 4(1+\xi_1^2)\boldsymbol{u}_{\chi_2} + \left(\frac{1+\xi_1^2}{\chi_2}\mathrm{A} + \frac{\kappa_2^2+\kappa_4^2}{4} + \mathcal{R}^2\right)\boldsymbol{u}.$$

8. Conclusion

This paper is a contribution to computer-aided algorithmic analysis of large nonlinear systems. We apply a certain nonlinear generalization of Lie's linear method for finding infinitesimal symmetries of PDE to a generalization of the CNLS system (2.1). The resulting large nonlinear systems of determining equations for nonclassical symmetries were analysed with considerable assistance from the software packages `diffgrob` and `rif`. Many genuinely new nonclassical symmetries were obtained. We found the corresponding reduced forms of the CNLS system. Our results show that the CNLS system is a rich source of such reductions.

Setting

$$\boldsymbol{U} = (U, Q, V, S)^{\mathrm{T}}, \qquad \mathrm{J}_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

the system (2.1) can be written as

$$-\mathrm{J}_0 \boldsymbol{U}_t + \nabla^2 \boldsymbol{U} + \mathcal{R}^2 \boldsymbol{U} = 0,$$

where $\mathcal{R}^2 = |\Psi|^2 + |\Phi|^2$. For the $\tau = 0$ reductions, it is convenient to use this notation due to the prominent appearance in the reduction equations of coupling matrices in a 4×4 representation of $\mathfrak{sl}(2,\mathbb{C})$. The $\tau = 0$ case detailed in this article yields a remarkable structure for the system of ISCs obtained as a result of solving for the nonclassical infinitesimals, and in the consequent reductions. We investigated ten separate solution families for the ISCs, yielding ten separate families of reductions. Remarkably in the $\tau = 0$ case, the expression $\mathcal{R}^2$ is an invariant, and the similarity variables obtained will also reduce equations of the form

$$-\mathrm{J}_0 \boldsymbol{U}_t + \nabla^2 \boldsymbol{U} + F(\mathcal{R}^2)\boldsymbol{U} = 0 \qquad (8.1)$$

for F arbitrary. In particular, one may take

$$F(\mathcal{R}^2) = \frac{\mathcal{R}^2}{1+\mathcal{R}^2},$$

which is the saturated form of the system. The further reduction of our reductions to systems of ordinary differential equations and their solutions will be examined elsewhere.

In this paper we analysed the determining system for nonclassical symmetries in the variables U, Q, V, S where $\Psi = U + \mathrm{i}Q, \Phi = V + \mathrm{i}S$. Alternatively and equivalently we could have analysed the systems in terms of $\Psi, \Psi^*, \Phi, \Phi^*$, regarding these as independent analytic variables. We believe that this may have led to some simplification of the calculations in the generic case.

It is an important and open problem not addressed in this paper to determine whether any of the nonclassical reductions yield solutions of the CNLS system (1.1) with physically significant properties. Such properties include being well-behaved globally, for example bounded without singularities and possessing suitable decay properties. Perturbations of such well-behaved solutions can be used to investigate interesting physical applications and typically involve a combination of analytic, numeric and symbolic techniques.

Lie's linear method has been successfully applied to many physically significant equations, and their symmetries and corresponding symmetry reduced equations have been determined. It might be expected that Lie's classical method becomes dramatically more difficult to apply as the number of independent and dependent variables increase. Indeed the determining systems for such high dimensional problems often contain thousands of PDE. Yet partly because of the linearity and sparsity, computer algebra packages have been able to explicitly solve many such systems.

While there has been significant progress and widely available software for the automatic analysis of linear overdetermined systems of PDE, few packages are available for nonlinear systems, and only limited automatic calculations have been done. These packages are still in the relatively early stages of development. The determining systems for the CNLS system were large and we believe provide a good test for the feasibility of the algorithms underpinning our packages. Despite the attendant case-splitting, we were able to use our computer algebra packages to carry out a significant portion of the analysis of these systems.

Currently, there is no algorithmic method known that simplifies the integration of the nonclassical infinitesimals, by exploiting known invariance properties of the original system of PDE. We mention the following approach to this important open problem. Extending the results of Lisle [47] and Fels and Olver [25,26], it should be possible to reformulate the nonclassical determining equations in terms of a moving frame which is invariant under the classical group. Theoretically, calculations in this frame should simplify the integration of the nonclassical determining equations, and we are exploring this possibility. Consider a family of differential equations with unspecified functions of the dependent variables in it, for example, heat equations with nonlinear diffusivity $K(u)$. An equivalence group of the family is a group of point transformations which preserves the family. Lisle [47] showed that use of a moving frame invariant under such an equivalence group, could lead to substantial simplification of the problem of integrating classical determining equations. Again we are exploring the possibility of extending such methods to the nonclassical case.

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