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Computation of the Maximal Degree of the Inverse of a Cubic Automorphism of the Affine Plane with Jacobian 1 via Gröbner Bases

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In this paper we propose to compute the maximal degree of the inverse of a cubic automorphism of the affine plane with Jacobian 1 via Gröbner Bases. This degree is equal to 9 and we give coefficients of the inverse.

1. Introduction

If k is any commutative ring, $k[X, Y]$ will denote the algebra of polynomials with coefficients in k in the indeterminates X, Y and $\mathbb{A}^2_k = \operatorname{Spec} k[X, Y]$ the affine plane over k. A k-endomorphism f of $\mathbb{A}^2_k$ will be identified with its coordinate functions $f = (f_1, f_2)$, where f_i $(i = 1, 2)$ belongs to $k[X, Y]$. We define the Jacobian of f by $\mathrm{Jac}(f) = \frac{\partial f_1}{\partial X}\frac{\partial f_2}{\partial Y} - \frac{\partial f_1}{\partial Y}\frac{\partial f_2}{\partial X}$ and the degree of f by $\deg(f) = \max_{1\le i\le 2} \deg(f_i)$.

Let d be a non-negative integer and f an endomorphism of $\mathbb{A}^2_{\mathbb{C}}$ whose degree is less than or equal to d. The Jacobian Conjecture in degree $d(\mathrm{CJ}(d))$ states that f is invertible if and only if its Jacobian is a non-zero constant.

Let C_d be the smallest integer C such that if k is a $\mathbb{Q}$-algebra and f a k-automorphism of $\mathbb{A}^2_k$ satisfying $\mathrm{Jac}(f) = 1$ and $\deg(f) \le d$, then we have $\deg(f^{-1}) \le C$.

Bass has proven the following result in Bass (1983):

Theorem 1.1. *The three following assertions are equivalent:*

(*i*) *$CJ(d)$ is true,*
(*ii*) *if k is any $\mathbb{Q}$-algebra and f any k-endomorphism of $\mathbb{A}^2_k$ whose degree is less than or equal to d, then f is invertible if and only if $\mathrm{Jac}(f)$ is an invertible element of $k[X, Y]$,*
(*iii*) *$C_d < \infty$.*

If k is a reduced $\mathbb{Q}$-algebra and f a k-automorphism of $\mathbb{A}^2_k$ satisfying $\mathrm{Jac}(f) = 1$ and $\deg(f) \le d$, it follows from a formula of Gabber (see Bass *et al.* (1982) and Cheng *et al.* (1994)) that $\deg f^{-1} = \deg f$. What happens if k is not reduced? Is it true that $C_d = d$ (see Question 2.19 of the paper by van den Essen (1991))?

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A negative answer to this question is given in Furter (to appear) where it is proven that $C_d \geq d+1$ as soon as $d \geq 3$. Also, Moh has proven that CJ(d) is true when $d \leq 100$ (see Moh (1983)). It then follows from Theorem 1.1 that C_d is finite for $d \leq 100$.

We could easily check that $C_1 = 1$. Theorem 2 of Furter (to appear) shows us that $C_2 = 2$. The purpose of this paper is to establish the following result:

THEOREM 1.2. $C_3 = 9$.

As far as we know, there is no explicit upper bound for C_d when $d \geq 4$ and there is not even a conjectured upper bound. An investigation of C_4 seems rather important to us in order to acquire some insight into the behaviour of C_d in general.

2. Computation of C_3

Let k be the algebra of polynomials with coefficients in $\mathbb{Q}$ in the indeterminates $a_1, a_2, a_3, b_1, b_2, b_3, b_4, c_1, c_2, c_3, d_1, d_2, d_3, d_4$ and let $f = (f_1, f_2)$ be the k-endomorphism of $\mathbb{A}_k^2$ whose coordinate functions are

$$\begin{cases} f_1 = X + a_3X^2 + a_2XY + a_1Y^2 + b_4X^3 + b_3X^2Y + b_2XY^2 + b_1Y^3, \\ f_2 = Y + c_3X^2 + c_2XY + c_1Y^2 + d_4X^3 + d_3X^2Y + d_2XY^2 + d_1Y^3. \end{cases}$$

Let $g = (g_1, g_2)$ be the formal inverse of f. The formal series g_1 and g_2 have expressions of the form

$$\begin{cases} g_1 = X + \sum_{(i,j)\in\mathbb{N}^2, i+j\geq 2} x_{i,j}X^iY^j, \\ g_2 = Y + \sum_{(i,j)\in\mathbb{N}^2, i+j\geq 2} y_{i,j}X^iY^j, \end{cases}$$

where $x_{i,j}, y_{i,j}$ belong to k.

The Jacobian of f is a polynomial with coefficients in k in the indeterminates X, Y. Its constant term is equal to 1 and we could check that its other non-trivial coefficients are equal to

$$\begin{cases} -3b_3d_4 + 3b_4d_3, \\ -6b_2d_4 + 6b_4d_2, \\ -9b_1d_4 - 3b_2d_3 + 3b_3d_2 + 9b_4d_1, \\ -6b_1d_3 + 6b_3d_1, \\ -3b_1d_2 + 3b_2d_1, \\ -3a_2d_4 + 2a_3d_3 - 2b_3c_3 + 3b_4c_2, \\ -6a_1d_4 - a_2d_3 + 4a_3d_2 - 4b_2c_3 + b_3c_2 + 6b_4c_1, \\ -4a_1d_3 + a_2d_2 + 6a_3d_1 - 6b_1c_3 - b_2c_2 + 4b_3c_1, \\ -2a_1d_2 + 3a_2d_1 - 3b_1c_2 + 2b_2c_1, \\ d_3 - 2a_2c_3 + 2a_3c_2 + 3b_4, \\ 2d_2 - 4a_1c_3 + 4a_3c_1 + 2b_3, \\ 3d_1 - 2a_1c_2 + 2a_2c_1 + b_2, \\ c_2 + 2a_3, \\ 2c_1 + a_2. \end{cases}$$

Let I be the ideal of k generated by the 14 polynomials given above.

Let us set $\bar{k} = k/I$. By reducing all the coefficients of f modulo I, we obtain a

$\bar{k}$-endomorphism of $\mathbb{A}^2_{\bar{k}}$ which we will denote by $\bar{f}$. Clearly, $\bar{f}$ is the generic cubic endomorphism with Jacobian 1 of the affine plane with the following meaning. Let A be any $\mathbb{Q}$-algebra and α be any cubic A-endomorphism of $\mathbb{A}^2_A$ with Jacobian 1. Up to an affine change of coordinates, we can always suppose that $\alpha(0) = 0$ and $\alpha'(0) = \mathrm{Id}$. Therefore, there exists a canonical algebra-homomorphism $\phi : \bar{k} \to A$ such that the A-endomorphism of $\mathbb{A}^2_A$ obtained by replacing the coefficients of $\bar{f}$ by their image under ϕ will be equal to α. As CJ(3) is true, the endomorphism $\bar{f}$ by their image under ϕ will be equal to α. As CJ(3) is true, the endomorphism $\bar{f}$ is an automorphism and we clearly have $C_3 = \deg(\bar{f})^{-1}$. Hence, the integer C_3 is the smallest integer C such that $x_{i,j}, y_{i,j}$ belongs to I as soon as $i + j > C$.

Using a computer, we found that the smallest integer N such that $x_{i,j}, y_{i,j}$ belongs to I as soon as $i + j = N$, is equal to 10. This encouraged us to believe that $C_3 = 9$ (and this already proved that $C_3 \geq 9$). Let h denote the k-endomorphism obtained from g by truncating its terms of degree bigger than or equal to 10. Then, to show that $C_3 = 9$, we only had to check that all coefficients of the endomorphism $f \circ h - \mathrm{Id}$ of $\mathbb{A}^2_k$ (whose degree is $9^3 = 729$) belong to I. Indeed, denoting by $\bar{h} = (\overline{h_1}, \overline{h_2})$ the $\bar{k}$-endomorphism of $\mathbb{A}^2_{\bar{k}}$ obtained by reducing the coefficients of h modulo I, the latter fact is equivalent to saying that the endomorphism $\bar{f} \circ \bar{h} - \mathrm{Id}$ of $\mathbb{A}^2_{\bar{k}}$ is identically zero, which is well known to ensure that $\bar{h} = (\bar{f})^{-1}$.

All computations were done using the computer algebra system AXIOM (see Jenks and Sutor (1983)).

3. Inversion Formula

Let us endow $k = \mathbb{Q}[a_1, \ldots, d_4]$ with the total degree-inverse lexicographical order (see Davenport *et al.* (1993)) for the following order of the indeterminates:

$$a_1 < a_2 < a_3 < c_1 < c_2 < c_3 < b_1 < b_2 < b_3 < b_4 < d_1 < d_2 < d_3 < d_4.$$

Considering the automorphism $(Y, X) \circ \bar{f} \circ (Y, X)$, one could easily show that the coefficient of X^iY^j in $\overline{h_2}$ is obtained from the coefficient of X^jY^i in $\overline{h_1}$ by replacing $a_1, a_2, a_3, c_1, c_2, c_3, b_1, b_2, b_3, b_4, d_1, d_2, d_3, d_4$ by $c_3, c_2, c_1, a_3, a_2, a_1, d_4, d_3, d_2, d_1, b_4, b_3, b_2,$

Coefficients of degree 2

coefficient of X^2	$\frac{1}{2}c_2$
coefficient of XY	$2c_1$
coefficient of Y^2	$-a_1$

Coefficients of degree 3

coefficient of X^3	$\frac{1}{2}b_4 + \frac{1}{2}d_3$
coefficient of X^2Y	d_2
coefficient of XY^2	$-\frac{1}{2}b_2 + \frac{3}{2}d_1$
coefficient of Y^3	$-b_1$

Coefficients of degree 4

coefficient of X^4	$\frac{1}{8}c_2b_4 - \frac{1}{2}c_3d_2 + \frac{3}{8}c_2d_3 - \frac{1}{2}c_1d_4$
coefficient of X^3Y	$c_1b_4 - 2c_3d_1 + c_1d_3$
coefficient of X^2Y^2	$-\frac{3}{2}(a_1b_4 + c_2d_1 + a_1d_3)$
coefficient of XY^3	$\frac{1}{3}c_1b_2 - 3c_1d_1 - \frac{4}{3}a_1d_2$
coefficient of Y^4	$c_1b_1 + \frac{1}{4}a_1b_2 - \frac{1}{4}a_1d_1$

Coefficients of degree 5

coefficient of X^5	$\frac{3}{4}b_4^2 + \frac{1}{4}d_3^2 + \frac{3}{4}b_3d_4 - \frac{1}{4}d_2d_4$
coefficient of X^4Y	$\frac{3}{4}b_3b_4 + \frac{1}{4}b_3d_3 + \frac{3}{4}d_2d_3 + 2b_2d_4 - \frac{3}{4}d_1d_4$
coefficient of X^3Y^2	$\frac{1}{2}b_2b_4 - \frac{3}{2}b_3d_2 + \frac{1}{2}d_2^2 + 2b_2d_3 + \frac{3}{2}d_1d_3 + 6b_1d_4$
coefficient of X^2Y^3	$-\frac{3}{2}b_1b_4 + 2d_1d_2 + \frac{3}{2}b_1d_3$
coefficient of XY^4	$-\frac{3}{4}b_1b_3 + \frac{3}{2}d_1^2 - \frac{1}{4}b_1d_2$
coefficient of Y^5	$-\frac{1}{4}b_1b_2 - \frac{3}{4}b_1d_1$

Coefficients of degree 6

coefficient of X^6	$\frac{1}{8}c_2b_4^2 - \frac{1}{4}c_3d_2d_3 + \frac{1}{4}c_2d_3^2 + \frac{1}{2}c_1b_4d_4 + \frac{7}{4}c_3d_1d_4 - \frac{5}{8}c_2d_2d_4$
coefficient of X^5Y	$\frac{3}{2}c_1b_4^2 - 5c_3d_1d_3 + \frac{7}{4}c_2d_2d_3 - 2c_1d_3^2 + 12c_1b_3d_4 + 33a_1b_4d_4$ $-\frac{51}{4}c_2d_1d_4 + \frac{57}{2}c_1d_2d_4 + 19a_1d_3d_4$
coefficient of X^4Y^2	$-\frac{15}{4}a_1b_4^2 - \frac{5}{2}c_1b_3d_3 - \frac{25}{4}c_2d_1d_3 + \frac{35}{4}c_1d_2d_3 + \frac{15}{2}a_1d_3^2$ $+ 15c_1b_2d_4 + 5a_1b_3d_4 + \frac{75}{4}c_1d_1d_4 - \frac{15}{4}a_1d_2d_4$
coefficient of X^3Y^3	$-\frac{5}{3}a_1b_3b_4 - \frac{85}{21}c_1d_2^2 - \frac{145}{63}c_1b_2d_3 - \frac{5}{7}a_1b_3d_3 + \frac{115}{21}c_1d_1d_3$ $-\frac{215}{63}a_1d_2d_3 + \frac{5}{7}c_1b_1d_4 - \frac{55}{7}a_1b_2d_4 + \frac{40}{7}a_1d_1d_4$
coefficient of X^2Y^4	$-\frac{5}{4}a_1b_2b_4 - \frac{5}{6}c_1b_2d_2 + \frac{10}{3}a_1b_3d_2 - 5c_1d_1d_2 - \frac{5}{12}a_1d_2^2$ $-\frac{5}{2}c_1b_1d_3 - \frac{55}{12}a_1b_2d_3 - 5a_1d_1d_3 - \frac{45}{5}a_1b_1d_4$
coefficient of XY^5	$\frac{3}{23}c_1b_1b_3 + 3a_1b_1b_4 - \frac{3}{2}c_1d_1^2 + c_1b_1d_2 - a_1d_1d_2$
coefficient of Y^6	$\frac{1}{3}c_1b_1b_2 + \frac{1}{4}a_1b_1b_3 + c_1b_1d_1 - \frac{1}{4}a_1d_1^2 + \frac{1}{6}a_1b_1d_2$

Coefficients of degree 7

coefficient of X^7	$\frac{5}{72}d_3^3 + \frac{3}{8}b_3b_4d_4 - \frac{5}{12}b_3d_3d_4 - \frac{11}{24}d_2d_3d_4 + 2b_2d_4^2 + \frac{27}{8}d_1d_4^2$
coefficient of X^6Y	$\frac{7}{24}d_2d_3^2 + \frac{21}{8}b_2b_4d_4 - 4d_2^2d_4 - \frac{1}{4}b_2d_3d_4 + 12d_1d_3d_4 + 18b_1d_4^2$
coefficient of X^5Y^2	$\frac{49}{24}d_1d_3^2 + \frac{113}{24}b_2b_3d_4 - \frac{75}{4}b_1b_4d_4 + \frac{53}{6}b_2d_2d_4$ $+\frac{7}{4}d_1d_2d_4 - \frac{43}{4}b_1d_3d_4$
coefficient of X^4Y^3	$\frac{5}{36}d_1d_2d_3 - \frac{65}{18}b_1d_3^2 + \frac{35}{72}b_2^2d_4 + \frac{35}{3}b_1b_3d_4 + \frac{305}{8}d_1^2d_4 + \frac{445}{12}b_1d_2d_4$

(cont).

Coefficients of degree 7 (Continued)

coefficient of X^3Y^4	$\frac{5}{4}b_1b_3d_3 + \frac{105}{8}d_1^2d_3 + 5b_1d_2d_3 + \frac{75}{8}b_1b_2d_4 + \frac{135}{4}b_1d_1d_4$
coefficient of X^2Y^5	$-\frac{7}{24}b_2^2d_2 + \frac{63}{8}d_1^2d_2 - \frac{5}{4}b_1d_2^2 + 2b_1b_2d_3 + \frac{39}{2}b_1d_1d_3 + \frac{27}{2}b_1^2d_4$
coefficient of XY^6	$-\frac{5}{8}b_1b_2b_3 + \frac{3}{8}b_1^2b_4 + \frac{21}{8}d_1^3 - \frac{3}{8}b_1b_2d_2 - \frac{3}{2}b_1^2d_3$
coefficient of Y^7	$-\frac{1}{8}b_1b_2^2 - \frac{9}{8}b_1d_1^2 - \frac{3}{4}b_1^2d_2$

Coefficients of degree 8

coefficient of X^8	$-\frac{1}{144}c_2d_3^3 - \frac{23}{192}c_2d_2d_3d_4 + \frac{29}{48}c_1d_3^2d_4 - \frac{11}{16}c_1b_3d_4^2 - \frac{33}{16}a_1b_4d_4^2$ $+\frac{147}{64}c_2d_1d_4^2 - \frac{31}{8}c_1d_2d_4^4 - \frac{11}{8}a_1d_3d_4^2$
coefficient of X^7Y	$-\frac{1}{9}c_1d_3^3 - \frac{11}{6}c_1b_3d_3d_4 - \frac{23}{12}c_1d_2d_3d_4 - 3a_1d_3^2d_4 + \frac{11}{2}c_1b_2d_4^2$ $+\frac{81}{4}c_1d_1d_4^2 + 9a_1d_2d_4^2$
coefficient of X^6Y^2	$\frac{7}{18}a_1d_3^3 + \frac{21}{4}c_1b_2d_3d_4 + \frac{203}{12}a_1b_3d_3d_4 + \frac{35}{24}a_1d_2d_3d_4$ $-\frac{189}{4}c_1b_1d_4^2 - \frac{203}{4}a_1b_2d_4^2 - \frac{189}{8}a_1d_1d_4^2$
coefficient of X^5Y^3	$\frac{7}{18}a_1d_2d_3^2 + 21c_1b_2d_2d_4 + \frac{21}{2}a_1d_2^2d_4 - 63c_1b_1d_3d_4$ $+\frac{203}{12}a_1b_2d_3d_4 - 35a_1d_1d_3d_4 - \frac{609}{4}a_1b_1d_4^2$
coefficient of X^4Y^4	$\frac{35}{36}a_1d_1d_3^2 - \frac{315}{4}c_1b_1b_3d_4 - \frac{2975}{144}a_1b_2b_3d_4 - \frac{805}{16}a_1b_1b_4d_4$ $-\frac{945}{4}c_1d_1^2d_4 - \frac{315}{2}c_1b_1d_2d_4 - \frac{2905}{72}a_1b_2d_2d_4$ $-\frac{245}{3}a_1d_1d_2d_4 - \frac{875}{24}a_1b_1d_3d_4$
coefficient of X^3Y^5	$\frac{21}{2}c_1b_1d_2d_3 + \frac{217}{36}a_1d_1d_2d_3 + \frac{581}{18}a_1b_1d_3^2 - \frac{7}{9}a_1b_2^2d_4 + \frac{7}{3}a_1b_1b_3d_4$ $-\frac{189}{2}c_1b_1d_1d_4 - \frac{217}{4}a_1d_1^2d_4 - \frac{581}{6}a_1b_1d_2d_4$
coefficient of X^2Y^6	$-\frac{21}{4}c_1b_1b_2d_3 - \frac{7}{36}a_1b_2^2d_3 - \frac{63}{4}a_1b_1b_3d_3 - \frac{77}{24}a_1b_1d_2d_3$ $+\frac{189}{4}c_1b_1^2d_4 + 49a_1b_1b_2d_4 + \frac{231}{8}a_1b_1d_1d_4$
coefficient of XY^7	$-3c_1b_1b_2d_2 - \frac{1}{9}a_1b_2^2d_2 - \frac{11}{6}a_1b_1d_2^2 + 9c_1b_1^2d_3$ $-\frac{23}{12}a_1b_1b_2d_3 + \frac{11}{2}a_1b_1d_1d_3 + \frac{81}{4}a_1b_1^2d_4$
coefficient of Y^8	$\frac{9}{8}c_1b_1^2b_3 + \frac{5}{32}a_1b_1b_2b_3 + \frac{51}{32}a_1b_1^2b_4 + \frac{27}{8}c_1b_1d_1^2 - \frac{3}{8}a_1d_1^3$ $+\frac{9}{4}c_1b_1^2d_2 + \frac{5}{16}a_1b_1b_2d_2 + \frac{3}{4}a_1b_1d_1d_2 + \frac{15}{16}a_1b_1^2d_3$

Coefficients of degree 9

coefficient of X^9	$-\frac{131}{144}d_2^2d_4^2 - \frac{131}{288}b_2d_3d_4^2 + \frac{131}{48}d_1d_3d_4^2 + \frac{131}{32}b_1d_4^3$
coefficient of X^8Y	$\frac{131}{32}b_2b_3d_4^2 - \frac{1179}{32}b_1b_4d_4^2 + \frac{131}{16}b_2d_2d_4^4 - \frac{393}{16}b_1d_3d_4^2$
coefficient of X^7Y^2	$-\frac{131}{32}b_1d_3^2d_4 - \frac{131}{32}b_2^2d_4^2 + \frac{393}{32}b_1b_3d_4^2 + \frac{393}{32}b_1d_2d_4^2$
coefficient of X^6Y^3	$\frac{917}{12}b_1b_3d_3d_4 + \frac{917}{24}b_1d_2d_3d_4 - \frac{917}{4}b_1b_2d_4^2 - \frac{2751}{8}b_1d_1d_4^2$
coefficient of X^5Y^4	$\frac{917}{8}b_1d_2^2d_4 + \frac{917}{16}b_1b_2d_3d_4 - \frac{2751}{8}b_1d_1d_3d_4 - \frac{8253}{16}b_1^2d_4^2$

(cont).

Coefficients of degree 9 (Continued)

coefficient of X^4Y^5	$\frac{8253}{32}b_1^2b_4d_4 + \frac{8253}{32}d_1^3d_4 + \frac{8253}{32}b_1d_1d_2d_4 + \frac{8253}{32}b_1^2d_3d_4$
coefficient of X^3Y^6	$\frac{917}{96}b_1^2d_3^2 + \frac{917}{96}b_1b_2^2d_4 - \frac{917}{32}b_1^2b_3d_4 - \frac{917}{32}b_1^2d_2d_4$
coefficient of X^2Y^7	$-\frac{131}{4}b_1^2b_3d_3 - \frac{131}{8}b_1^2d_2d_3 + \frac{393}{4}b_1^2b_2d_4 + \frac{1179}{8}b_1^2d_1d_4$
coefficient of XY^8	$-\frac{131}{16}b_1^2d_2^2 - \frac{131}{32}b_1^2b_2d_3 + \frac{393}{16}b_1^2d_1d_3 + \frac{1179}{32}b_1^3d_4$
coefficient of Y^9	$-\frac{131}{64}(b_1^3b_4 + b_1d_1^3 + b_1^2d_1d_2 + b_1^3d_3)$

b_1, respectively. Now we give the coefficients of $\overline{h_1}$, or, to be more precise, the coefficients of h_1 reduced modulo the Gröbner basis of I (see Davenport *et al.* (1993)).

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