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Implicitization of Rational Parametric Surfaces

GEORGE FIX, CHIH-PING HSU† AND TIE LUO‡

Box 19408, Mathematics Department
The University of Texas at Arlington
Arlington, TX 76019, U.S.A.

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A generalized projective implicitization theorem is presented that can be used to solve the implicitization of rational parametric curves and surfaces in an affine space. The Groebner bases technique is used to implement the algorithm. The algorithm has the advantages that it can handle base points in a parametrization, and no extra factors will be introduced into an implicit equation. The complexity of the algorithm in terms of the degrees of the polynomials in the Groebner basis is better than the existing method.

1. Introduction

In this paper we consider the implicitization problem for rational parametric surfaces. The parametric form has many advantages in geometric modeling. An implicit equation is useful in some situations (for example, in finding the intersection of two parametrized surfaces or to determine whether a given point lies on a surface). Let $S \subset \mathbb{C}^3$ be a surface parametrized by a map

$$F_a(s,t) = (f_1(s,t),\ f_2(s,t),\ f_3(s,t)),$$

where f_i are polynomials in s and t. The process of finding an implicit equation of S is called implicitization. The implicitization would find an equation $f(x,y,z) = 0$ such that the zero locus of the equation,

$$W = \{(x,y,z) \mid f(x,y,z) = 0\},$$

is the smallest subset $W \subset \mathbb{C}^3$ that contains the image S of the parametrization. A common approach for implicitization is to consider the variety V in the affine space $\mathbb{C}^{2+3}$ which is defined by the equations

$$x_i - f_i(s,t) = 0, \qquad i = 1,2,3. \tag{1.1}$$

The variety V is interpreted as the graph of the parametrization $F_a(s,t)$. An implicit equation describes the algebraic relation among the coordinates x, y, and z of the surface

† E-mail: b227hsu@utamat.uta.edu
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S. Therefore, the implicitization can be thought of as finding the algebraic combinations of the parametric equations (1.1) which eliminate the variables s and t.

Several methods are known for implicitization. Sederberg (1984) considered a polynomial implicitization algorithm based on the method of resultants for variables elimination. Kalkbrener (1990) solved implicitization of the rational parametrization that required decompositions and calculations of the greatest common divisor of multivariate polynomials. To deal with the base points, Manocha and Canny (1992a, b) added a perturbation variable to the parametric equations and then applied the resultant method to eliminate the parameters. The resulting polynomial may contain extraneous components. One major difficulty of this algorithm is to factor out the right component that contains the surface S.

The central issue in implicitization is that of elimination of the parameters. In the literatures the methods of using resultant, Groebner bases, and characteristic set are often considered. The resultant method has been considered by various authors (Sederberg, 1984; Manocha and Canny, 1992a, b). The approach based on the Groebner basis was first introduced by Buchberger in 1965. The details of the method can be found in Buchberger (1985), Cox *et al.* (1992) and the references therein. In addition, an introduction to these elimination methods can be found in Kapur (1992). The curves and surfaces are geometric objects and implicitization involves algebraic manipulations of the parametric equations. Thus, the concepts of varieties and ideals from algebraic geometry are drawn naturally into the implicitization. To apply the Groebner basis technique, let us consider the ideal I generated by the parametric equations (1.1),

$$I = \langle x_1 - f_1,\ x_2 - f_2,\ x_3 - f_3 \rangle \subset k[s, t, x_1, x_2, x_3].$$

First, a lexicographical monomial ordering is chosen so that the parameters s and t are greater than any other variable x_i. Then we compute a Groebner basis G for the ideal I, and define I_2 to be the second elimination ideal. The ideal I_2 has a Groebner basis G_2 which consists of the polynomials in G not involving the parameters s and t. It follows that I_2 represents all the algebraic consequences after eliminating the parameters s and t from the parametric equations. Therefore, I_2 contains an implicit equation of S.

For a surface S with rational parametrization,

$$F_a(s,t) = \left(\frac{f_1(s,t)}{w(s,t)},\ \frac{f_2(s,t)}{w(s,t)},\ \frac{f_3(s,t)}{w(s,t)} \right),$$

where f_i and w are polynomials in s and t, we consider the equations

$$F_i = w\, x_i - f_i, \qquad i = 1, 2, 3, \tag{1.2}$$

for the implicitization. Note that the variety U determined by the equations after eliminating s and t from (1.2) may be unreasonably larger than the original surface S. This is because $w = 0$ is allowed in F_i but $F_a(s,t)$ is not defined on the locus $w = 0$. Hence, before applying the Groebner basis method, we need a set of polynomials $\{g_1, \ldots, g_n\}$ which guarantees, after eliminating s and t, $U = S$. If equality is not possible, then we would like to have a U which is the smallest variety containing S. Kalkbrener further suggested that a control polynomial $1 - w(s,t) \cdot y$ should be added to the parametric equations to avoid the zeros of the denominators (Cox *et al.*, 1992; Hoffmann, 1993).

Since a rational variety can be identified as an affine portion of a projective variety, it is natural to solve the implicitization problem in the projective space. From the rational parametrization $F_a(s,t)$, we have a projective parametrization $F : \mathbb{P}^2 \to \mathbb{P}^3$ defined by

$$F(s,t,1) = \left[\frac{f_1(s,t)}{w(s,t)}, \frac{f_2(s,t)}{w(s,t)}, \frac{f_3(s,t)}{w(s,t)}, 1\right],$$

which, after homogenization, is equivalent to

$$F(s,t,u) = [f_1^h(s,t,u),\ f_2^h(s,t,u),\ f_3^h(s,t,u),\ w^h(s,t,u)].$$

When applying the Groebner basis method to the equations, we obtain a homogeneous implicit equation $g^h = 0$. After dehomogenization procedure with respect to the last variable of $\mathbb{P}^3$, we have a desired implicit equation $g = 0$ in the affine space $\mathbb{C}^3$.

A projective implicitization theorem (Theorem 2.5) is presented in Section 2 which can solve the problem caused by the base points. The proof of the theorem relies on Groebner basis theory, and Buchberger's Groebner basis algorithm is used to implement the algorithm for implicitization. The theorem is a natural generalization of that without base point, which can be found in Cox *et al.* (1992). Our method of homogenization is different from those known in the literature. However, we want to leave open to discussion whether our method improves the efficiency of computation.

Throughout the discussion, we use bracket $[x,y,z,w]$ to represent points in projective space $\mathbb{P}^3$ and parenthesis (x,y,z) to represent points in affine space $\mathbb{C}^3$.

2. Implicitization

Let $V \subset \mathbb{P}^n$ be a projective variety parametrized by the map $F : \mathbb{P}^m \to \mathbb{P}^n$,

$$F(x_0,\ldots,x_m) = [f_0(x_0,\ldots,x_m),\ldots,f_n(x_0,\ldots,x_m)],$$

where f_i are homogeneous polynomials in x_i of equal degree. If the map F has no base points, we can define the following mth elimination ideal

$$I_{m+1} = I \cap k[y_0,\ldots,y_n], \qquad \text{where } I = \langle y_0 - f_0,\ldots,y_n - f_n\rangle.$$

Note that I_{m+1} contains all the consequences of eliminating $x_0,\ldots,x_m$ from the parametric equations

$$y_i - f_i = 0, \qquad i = 0,\ldots,n.$$

Furthermore, we have $F(\mathbb{P}^m) = V(I_{m+1})$.

When F has base points, which by definition are points $p \in \mathbb{P}^m$ such that $F(p) = 0$, then the above statement is not true in general as illustrated in the following example:

EXAMPLE 2.1. Let $F : \mathbb{P}^2 \to \mathbb{P}^3$ be a parametrization of a plane in $\mathbb{P}^3$ defined by

$$F(s,t,u) = [s^2 - t^2,\ u^2,\ s\,u,\ s^2 - t^2 + u^2 + s\,u].$$

Note that $B = \{[1,\pm 1,0]\}$ is the set of base points, that is, $F(B) = [0,0,0,0] \notin \mathbb{P}^3$. It can be shown that a Groebner basis G of the ideal

$$I = \langle x - s^2 + t^2,\ y - u^2,\ z - s\,u,\ w - s^2 + t^2 - u^2 - s\,u\rangle$$

with lexicographic order $s > t > u > x > y > z > w$ is

$$\begin{aligned} G = \{ & -w + x + y + z,\ -u^2 + y,\ t^2\,y + w\,y - y^2 - y\,z - z^2, \\ & -u\,t^2 + w - y - z + s\,z,\ -s\,y + u\ z,\ s\,u - z,\ s^2 - t^2 - w + y + z\}. \end{aligned}$$

Thus $I_3 = \langle x+y+z-w \rangle$, and an implicit equation for the variety which covers the image of the parametrization F is $x+y+z-w=0$. The point $p=[0,0,1,1] \in V(I_3)$ but $p \notin F(\mathbb{P}^2 \setminus B)$. Hence we have $F(\mathbb{P}^2 \setminus B) \subsetneq V(I_3)$.

The following example illustrates that the equality can be achieved in some cases.

EXAMPLE 2.2. Consider a variety parametrized by a map $F : \mathbb{P}^2 \to \mathbb{P}^3$ defined by $F(s,t,u) = [s,\ s,\ t,\ t]$. A Groebner basis for the ideal

$$I = \langle x-s,\ y-s,\ z-t,\ w-z \rangle$$

with lexicographic order $s > t > x > y > z > w$ is

$$G = \{x-y,\ z-w,\ w-t,\ y-s\}.$$

Observe that

$$F(\mathbb{P}^2) \subset V(x-y,\ z-w) = \text{line in } \mathbb{P}^3.$$

However, the base point set B of F is $\{[0,0,1]\}$. Thus $F(\mathbb{P}^2 \backslash B) = V(x-y,\ z-w)$.

Notice that in Example 2.1, dim $F(\mathbb{P}^2) = 2 =$ dim $\mathbb{P}^2$ and in Example 2.2, dim $F(\mathbb{P}^2) = 1 < 2 = \dim \mathbb{P}^2$. These examples suggest that if the set of base points B is empty, then $F(\mathbb{P}^2) = V(I_3)$. Otherwise, the best we could hope for is $F(\mathbb{P}^2 \setminus B) \subseteq V(I_3)$.

In the proof of the projective implicitization theorem (Cox, 1992), an ideal J generated by bihomogeneous polynomials, $J = \langle y_i f_j - y_j f_i \rangle$, is used to prove the equality $F(\mathbb{P}^m) = V(I_{m+1})$. Such an approach fails when F has base points. This is due to the following: Let p be a base point. Then $p \times \mathbb{P}^n \subset V(J)$ and the projection $\pi(V(J)) = \mathbb{P}^n$, which is too big to be the smallest variety containing $F(\mathbb{P}^m \setminus B)$.

The following lemma in Cox *et al.* (1992) is necessary for the main theorem.

LEMMA 2.3. **(Polynomial Implicitization)** *Let k be the field $\mathbb{C}$ or $\mathbb{R}$, and $F : k^m \to k^n$ be the function determined by the polynomials parametrization*

$$\begin{cases} y_1 = f_1(x_1,\ldots,x_m) \\ \quad\vdots \\ y_n = f_n(x_1,\ldots,x_m) \end{cases}$$

where $f_1,\ldots,f_n \in k[x_1,\ldots,x_m]$. Let $I = \langle y_1 - f_1,\ldots,y_n - f_n \rangle \subset k[x_1,\ldots,x_m,y_1,\ldots,y_n]$ and let $I_m = I \cap k[y_1,\ldots,y_n]$ be the mth elimination ideal. Then $V(I_m)$ is the smallest variety in k^n containing $F(k^m)$.

The idea of the proof of Lemma 2.3 is to look at the graph V of F defined by $\langle y_1 - f_1,\ldots,y_n - f_n \rangle$ in $k^m \times k^n$. It is easy to see that $F(k^m) = \pi_m(V) \subset V(I_m)$, where π_m is the projection from $k^m \times k^n$ to k^n. To show that $V(I_m)$ is the smallest variety containing $F(k^m)$ requires the checking of equality $V(I_m) = V(I(F(k^m)))$. This can be achieved by Hilbert's Nullstellensatz.

If the variety $V(I_m)$ in Lemma 2.3 is not equal to $F(k^m)$ and k is algebraically closed, then the missing points in $V(I_m) \setminus F(k^m)$ are unions of some lower dimensional varieties. In addition, there may be some curves on a surface or some points on a curve. If k is not algebraically closed then $V(I_m)$ can be much larger than $F(k^m)$ even for a simple parametrization. For example, let $F(s,t) = (s^2,\ t^2,\ s\ t) \subset \mathbb{R}^3$ then $V(I_2) = V(z^2 - xy)$. However, $F(s,t)$ covers only half of $V(I_2)$.

Lemma 2.4. *The variety $V(I_m)$ is irreducible.*

Lemma 2.4 has an easier interpretation from the geometry point of view. Suppose V is not irreducible and $V = V_1 \cup V_2$ where V_1 and V_2 are proper subsets of V. Let $F^{-1}(V_1)$ and $F^{-1}(V_2)$ be the preimages of V_1 and V_2, then $k^m = F^{-1}(V_1) \cup F^{-1}(V_2)$. Hence k^m is not irreducible, a contradiction.

We now prove the main theorem which takes the case of base points into consideration.

Theorem 2.5. *Let B be the set of base points of a parametrization $F : \mathbb{P}^m(k) \to \mathbb{P}^n(k)$ defined by*

$$F(x_0, \ldots, x_m) = [f_0(x_0, \ldots, x_m), \ldots, f_n(x_0, \ldots, x_m)],$$

where f_i are homogeneous polynomials of the same degree in $k[x_0, \ldots, x_m]$. Then $F(\mathbb{P}^m \setminus B) \subseteq V(I_{m+1})$, where I_{m+1} is the $(m+1)$th elimination ideal of the ideal $I = \langle y_0 - f_0, \ldots, y_n - f_n\rangle$, and $V(I_{m+1})$ is the smallest variety that contains $F(\mathbb{P}^m \setminus B)$.

Proof. To prove the theorem, we use the natural correspondence between the affine variety V_a in k^{n+1} and the projective variety V in $\mathbb{P}^n$. Thus we can use the affine version of the theorem to prove the projective version of the theorem. A set of homogeneous polynomials of the same degree $\{f_0(x_0, \ldots, x_m), \ldots, f_n(x_0, \ldots, x_m)\}$ defines a projective map $F : \mathbb{P}^m(k) \to \mathbb{P}^n(k)$ and also an affine map $F_a : k^{m+1} \to k^{n+1}$. Note that F_a is everywhere defined. Then we have (by Lemma 2.3)

$$F_a(k^{m+1}) \subseteq V_a(I_{m+1})$$

where I_{m+1} is the $(m+1)$th elimination ideal of $I = \langle y_0 - f_0, \ldots, y_n - f_n\rangle$. Furthermore, $V_a(I_{m+1})$ is the smallest variety in k^{n+1} containing $F_a(k^{m+1})$. Let B_a be the affine cone over B. There are two natural correspondences α and β such that

$$\begin{tikzcd} k^{m+1}\backslash B_a \arrow[r, "F_a"] \arrow[d, "\alpha"'] & k^{n+1}\backslash\{0\} \arrow[d, "\beta"'] \\ \mathbb{P}^m\backslash B \arrow[r, "F"] & \mathbb{P}^n \end{tikzcd}$$

We now claim that I_{m+1} is a homogeneous ideal. Since $y_i - f_i$ are not homogeneous we will introduce weights on the variables $x_0, \ldots, x_m, y_0, \ldots, y_n$. Suppose f_i have total degree d, we arrange that each x_i has weight 1 and each y_i has weight d. Then a monomial $x^\gamma y^\delta$ has weight $|\gamma| + d|\delta|$ and a polynomial $f \in k[x_0, \ldots, x_m, y_0, \ldots, y_n]$ is weighted homogeneous provided every monomial has the same weight. Now we may apply Buchberger's algorithm to compute a Groebner basis G for I with respect to lexicographic order. Since the S-polynomial of two homogeneous polynomials is homogeneous, then G consists of weighted homogeneous polynomials and $G \cap k[y_0, \ldots, y_n]$ is a Groebner basis of I_{m+1}. Thus I_{m+1} is a weighted homogeneous ideal. Furthermore, each y_i has the same weight d. Hence, I_{m+1} is a homogeneous ideal. This proves our claim.

The variety $V(I_{m+1})$ is well defined in $\mathbb{P}^n$ with

$$\beta(V_a(I_{m+1}) \setminus \{0\}) = V(I_{m+1}).$$

For each point $p \in \beta(V_a(I_{m+1})\setminus\{0\})$, there is a nonzero q in $V_a(I_{m+1})$ such that $p = \beta(q)$. Hence p must be in $V(I_{m+1})$. Conversely for $p \in V(I_{m+1})$, p is nonzero. Therefore, by the definition of β, p must be in $\beta(V_a(I_{m+1}) \setminus \{0\})$. This implies that $V(I_{m+1})$ is the smallest variety in $\mathbb{P}^n$ that contains $F(\mathbb{P}^m \setminus B)$. Otherwise, through the map β, the affine variety $V_a(I_{m+1})$ would not be the smallest variety in k^n containing $F_a(\mathbb{P}^m \setminus B_a)$, this completes the proof. $\square$

By similar arguments, we have the following result which shows an implicit equation contains no extraneous factors in the case I_{m+1} is generated by one polynomial.

Corollary 2.6. *The projective variety $V(I_{m+1})$ is irreducible.*

The following example illustrates implicitization by using the main theorem.

Example 2.7. Consider a map $F : \mathbb{P}^2(\mathbb{C}) \to \mathbb{P}^3(\mathbb{C})$ defined by

$$F_a(s,t,u) = [s^2 - t^2 - u^2,\ 2\,s\,u,\ 2\,s\,t,\ s^2 + t^2 + u^2].$$

Then $[0, \pm\, i, 1]$ are base points. Calculating a Groebner basis G of the ideal

$$I = \langle x - s^2 + t^2 + u^2,\ y - 2\,s\,u,\ z - 2\,s\,t,\ w - s^2 - t^2 - u^2\rangle$$

with lexicographic order $s > t > u > x > y > z > w$, we have

$$\begin{aligned} G = \{\, & w^2 - x^2 - y^2 - z^2, 2\,u^2\,y^2 - w\,y^2 + x\,y^2 + 2\,u^2\,z^2,\\ & -2\,u^2\,w - 2\,u^2\,x + y^2, t\,y - u\,z, -\left(2\,u^2 - w + x\right),\\ & y - 2\,t\,u\,z, -2\,t\,u\,w - 2\,t\,u\,x + y\,z, -2\,t^2 - 2\,u^2 + w - x,\\ & -t\,(-w - x) - s\,z, -u\,(-w - x) - s\,y, -(s\,w) + s\,x + u\,y + t\,z,\\ & -2\,s\,u + y, -2\,s\,t + z, 2\,s^2 - w - x\}. \end{aligned}$$

It follows that $I_3 = I \cap k[x,y,z,w] = \langle w^2 - x^2 - y^2 - z^2\rangle$. An implicit equation of the map is $w^2 - x^2 - y^2 - z^2 = 0$ and $V(I_3)$ is the smallest variety containing the image of the map F. If we dehomogenize$w^2 - x^2 - y^2 - z^2 = 0$ with respect to w we obtain $x^2 + y^2 + z^2 = 1$, which is the unit sphere in k^3.

The following example is provided to illustrate that the theorem can be employed to solve the problem of implicitization of rational parametric curves and surfaces.

Example 2.8. *Consider the following rational parametrization $F_a : \mathbb{R}^2 \to \mathbb{R}^3$ defined by*

$$F_a(s,t) = \left(\frac{s^2}{t}, \frac{t^2}{s}, s\right).$$

Note that F_a is undefined at point $(0,0)$. To find an implicit equation, we can consider the projective version of the map $F : \mathbb{P}^2 \to \mathbb{P}^3$ defined by

$$F(s,t,u) = [s^3,\ t^3,\ s^2 t,\ s\ t\ u].$$

Also note that $[0,0,1]$ is a base point of F. The Groebner basis G of the ideal

$$I = \langle x - s^3,\ y - t^3,\ z - s^2\ t,\ w - s\ t\ u\rangle$$

with lexicographic order $s > t > u > x > y > z > w$ *is*

$$\begin{aligned} G = \{ & -x^2\, y + z^3,\; w^3\, x - u^3\, z^3,\; w^3 - u^3\, x\, y,\; -t\, w^4 + u^4\, y\, z^2,\; -u\, x\, y + t\;\; w\;\; z, \\ & -t\, w\, x + u\, z^2,\; -w^2 + t\, u^2\, z,\; -t^2\, w^2 + u^2\, y\, z,\; w\, x\, y - t^2\, u\, z^2, \\ & -t^2\, u\, x + w\, z, -t^3 + y,\; s\, w - u\, z,\; t\, x - s\, z,\; s\, x\, y - t\, z^2,\; t^2\, w - s\, u\, y, \\ & -s\, t\, u + w,\; -s\, t^2\, x + z^2,\; s^2\, y - t^2\, z,\; -s^2\, t + z, -s^3 + x\} \end{aligned}$$

Hence, $I_3 = I \cap k[x,\; y,\; z,\; w] = \langle -x^2y - z^3\rangle$, *and an implicit equation for the variety is* $z^3 - x^2y = 0$.

The following example was considered in Manocha and Canny (1992) which required decomposition of the multivariate polynomial. Note that no extra factors occur in our approach.

EXAMPLE 2.9. *Consider the rational parametrization* F_a *from* $\mathbb{R}^2$ *to* $\mathbb{R}^3$ *defined by*

$$F_a(s,t) = \left(\frac{s\; t^2 - t}{s\; t^2},\; \frac{s\; t + s}{s\; t^2},\; \frac{2\; s - 2\; t}{s\; t^2} \right).$$

Then the projective version of the map F_a *is* $F : \mathbb{P}^2 \to \mathbb{P}^3$ *defined by*

$$F(s,t,u) = [s\, t^2 - t\, u^2,\; s\, t\, u + s\, u^2,\; 2\, s\, u^2 - 2\, t\, u^2,\; s\, t^2].$$

The Groebner basis of the ideal

$$I = \langle x - s\, t^2 + t\, u^2,\; y - s\, t\, u - s\, u^2,\; z - 2\, s\, u^2 + 2\, t\, u^2,\; w - s\, t^2\rangle$$

with lexicographic order $s > t > u > x > y > z > w$ *is*

$$\begin{aligned} G = \{ & -4\, w\, x + 4\, x^2 - 8\, w\, y + 8\, x\, y + 4\, y^2 + 2\, w\, z - 4\, x\, z - 4\, y\, z + z^2, \\ & -4\, u^3\, w - 4\, w^2 + 4\, w\, x - 4\, w\, y + 4\, x\, y + 4\, y^2 - 2\, x\, z - 4\, y\, z + z^2, \\ & -2\, u^3\, x - 2\, u^3\, y + 2\, w\, y - 2\, x\, y + u^3\, z, -2\, t\, y - u(-2\, x - 2\, y + z), \\ & -2\, t\, w - 2\, u\, w + 2\, t\, x + 2\, u\, x + 2\, u\, y - t\, z - u\, z, -(t\, u^2) + w - x, \\ & -4\, u\, x\, y + 4\, s\, y^2 - 4\, u\, y^2 - 2\, s\, w\, z - 2\, u\, w\, z + 2\, u\, x\, z - 4\, s\, y\, z \\ & +4\, u\, y\, z + s\, z^2 - u\, z^2, \\ & 2\, s\, w + 2\, u\, w - 2\, s\, x - 2\, u\, x - 2\, u\, y + u\, z, 2\, s\, u^2 - 2\, w + 2\, x - z, \\ & -s\, t\, z - u(2\, u\, x - 2\, s\, y + 2\, u\, y + s\, z - u\, z), -2\, s\, t\, u - 2\, w + 2\, x + 2\, y - z, \\ & -s\, t^2 + w\}. \end{aligned}$$

Hence, $I_3 = I \cap k[x,y,z,w] = \langle -4\, w\, x + 4\, x^2 - 8\, w\, y + 8\, x\, y + 4\, y^2 + 2\, w\, z - 4\, x\, z - 4\, y\, z + z^2\rangle$, *and an implicit equation of the parametrization is* $-4\; x + 4\, x^2 - 8\; y + 8\, x\, y + 4\, y^2 + 2\; z - 4\, x\, z - 4\, y\, z + z^2 = 0$.

3. Conclusions

In the preceding section a generalized projective implicitization theorem is presented and is applied to find an implicit equation of a rationally parametrized curve or surface. The advantage of this algorithm is that it will give the smallest variety which contains the image of the parametrization without any extra factor when the base points appear in the parametrization (see Manocha and Canny, 1992a, b for example). Since the algorithm relies on the Groebner basis technique, there is a concern about the efficiency of calculating the Groebner basis for an ideal. However, as pointed out in Cox (1992), for most geometric problems the storage space and running time required by the construction of the Groebner basis appear to be more manageable than in the worst cases. Mayr and Meyer (1982) showed that the construction of Groebner basis for an ideal generated by polynomials of degree less than or equal to d would involve polynomials of degree proportional to 2^{2^d}. While the homogenization of a polynomial does not change the total degree, the method of introducing a control polynomial gives input polynomials of degree $d+1$.

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