GRÖBNER BASES, INITIAL IDEALS AND INITIAL ALGEBRAS

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ABSTRACT. We give an introduction to the theory of initial ideals and initial algebras with emphasis on the transfer of structural properties.

The notion of Gröbner basis of an ideal is the foundation of all efficient computations in algebraic geometry and commutative algebra. Highly sophisticated algorithms have been implemented in several, widely used computer programs.

However, Gröbner bases and their analogue for subalgebras are also important from a purely structural point of view. They allow us to find deformations of interesting, but "complicated" rings R to simpler objects R' that are defined by monomials and therefore accessible to combinatorial methods. See [4] for a paradigmatic case. In order to transfer the properties that have been found for R' back to R, one has to understand how R and R' are related. In this article we want to explain this relationship and to prove some of the basic results about the passage from R to R', or rather the other way round.

In [4] we have treated the subject in a similar manner. However, we hope that some readers will welcome a separate discussion that is independent from determinantal ideals and rings. Moreover, the material covered has been slightly expanded and some proofs are given in more detail.

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1. INITIAL VECTOR SPACES, IDEALS AND SUBALGEBRAS

Let us first recall the definitions and some important properties of Gröbner bases, monomial orders, initial ideals and initial algebras. For further information on the theory of Gröbner bases we refer the reader to the books by Eisenbud [8], Eisenbud et al. [9], Greuel and Pfister [12],

Kreuzer and Robbiano [14], Sturmfels [17] and Vasconcelos [18]. For the so-called Sagbi bases and initial algebras one should consult Conca, Herzog and Valla [7], Robbiano and Sweedler [15], and [17, Chapter 11]. Many applications of Gröbner bases are discussed in Buchberger and Winkler [6].

Throughout this section let K be a field, and let R be the polynomial ring $K[X_1, \ldots, X_n]$. A *monomial* (or power product) of R is an element of the form $X^\alpha = \prod_{i=1}^n X_i^{\alpha_i}$ with $\alpha \in \mathbb{N}^n$. A *term* is an element of the form λm where λ is a non-zero element of K and m is a monomial. Let $M(R)$ be the K-basis of R consisting of all the monomials of R. Every polynomial $f \in R$ can be written as a sum of terms. The only lack of uniqueness in this representation is the order of the terms. If we impose a total order on the set $M(R)$, then the representation is uniquely determined, once we require that the monomials are written according to the order, from the largest to the smallest. The set $M(R)$ is a semigroup (naturally isomorphic to $\mathbb{N}^n$) and a total order on the set $M(R)$ is not very useful unless it respects the semigroup structure.

Definition 1.1. A *monomial order* τ is a total order $<_\tau$ on the set $M(R)$ which satisfies the following conditions:

(a) $1 <_\tau m$ for all the monomials $m \in M(R) \setminus \{1\}$.
(b) If $m_1, m_2, m_3 \in M(R)$ and $m_1 <_\tau m_2$, then $m_1 m_3 <_\tau m_2 m_3$.

From the theoretical as well as from the computational point of view it is important that descending chains in $M(R)$ terminate:

Remark 1.2. A *monomial order* on the set $M(R)$ is a well-order, i.e. every non-empty subset of $M(R)$ has a minimal element. Equivalently, there are no infinite descending chains in $M(R)$.

This follows from the fact that every (monomial) ideal in R is finitely generated. Therefore a subset N of $M(R)$ has only finitely many elements that are minimal with respect to divisibility. One of them is the minimal element of N.

We list the most important monomial orders.

Example 1.3. For monomials $m_1 = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ and $m_2 = X_1^{\beta_1} \cdots X_n^{\beta_n}$ one defines

(a) the *lexicographic order* (Lex) by $m_1 <_{\text{Lex}} m_2$ iff for some k one has $\alpha_k < \beta_k$ and $\alpha_i = \beta_i$ for $i < k$;
(b) the *degree lexicographic order* (DegLex) by $m_1 <_{\text{DegLex}} m_2$ iff $\deg(m_1) < \deg(m_2)$ or $\deg(m_1) = \deg(m_2)$ and $m_1 <_{Lex} m_2$;
(c) the *(degree) reverse lexicographic order* (RevLex) by $m_1 <_{\text{RevLex}} m_2$ iff $\deg(m_1) < \deg(m_2)$ or $\deg(m_1) = \deg(m_2)$ and for some k one has $\alpha_k > \beta_k$ and $\alpha_i = \beta_i$ for $i > k$.

These three monomial orders satisfy $X_1 > X_2 > \cdots > X_n$. More generally, for every total order on the indeterminates one can consider the Lex, DegLex and RevLex orders extending the order of the indeterminates; just change the above definition correspondingly.

From now on we fix a monomial order τ on (the monomials of) R. Whenever there is no danger of confusion we will write $<$ instead of $<_\tau$. Every polynomial $f \neq 0$ has an unique representation

$$f = \lambda_1 m_1 + \lambda_2 m_2 + \cdots + \lambda_k m_k$$

where $\lambda_i \in K \setminus \{0\}$ and $m_1, \ldots, m_k$ are distinct monomials such that $m_1 > \cdots > m_k$. The *initial monomial* of f with respect to τ is denoted by $\mathrm{in}_\tau(f)$ and is, by definition, m_1. Clearly one has

$$\mathrm{in}_\tau(fg) = \mathrm{in}_\tau(f)\,\mathrm{in}_\tau(g) \tag{1}$$

and $\mathrm{in}_\tau(f + g) \leq \max_\tau\{\mathrm{in}_\tau(f), \mathrm{in}_\tau(g)\}$. For example, the initial monomial of the polynomial $f = X_1 + X_2 X_4 + X_3^2$ with respect to the Lex order is X_1, with respect to DegLex it is $X_2 X_4$, and with respect to RevLex it is X_3^2.

Given a K-subspace $V \neq 0$ of R, we define

$$M_\tau(V) = \{\mathrm{in}_\tau(f) : f \in V\}$$

and set

$$\mathrm{in}_\tau(V) = \text{ the } K\text{-subspace of } R \text{ generated by } M_\tau(V).$$

The space $\mathrm{in}_\tau(V)$ is called the *space of the initial terms of* V. Whenever there is no danger of confusion we suppress the reference to the monomial order and use the notation $\mathrm{in}(f)$, $M(V)$ and $\mathrm{in}(V)$.

Any positive integral vector $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$ induces a graded structure on R, called the *a-grading*. With respect to the a-grading the indeterminate X_i has degree $a(X_i) = a_i$. Every monomial X^α is a-homogeneous of a-degree $\sum \alpha_i a_i$, and the a-degree $a(f)$ of a non-zero polynomial $f \in R$ is the largest a-degree of a monomial in f. Then $R = \bigoplus_{i=0}^\infty R_i$ where R_i is the *a-graded component* of R of degree i, i.e. the span of the monomials of a-degree i. With respect to this decomposition R has the structure of a positively graded K-algebra [5, Section 1.5]. The elements of R_i are *a-homogeneous* of a-degree i. We say that a vector subspace V of R is *a-graded* if it is generated, as a vector space, by homogeneous elements. This amounts to the decomposition $V = \bigoplus_{i=0}^\infty V_i$ where $V_i = V \cap R_i$.

Proposition 1.4. *Let V be a K-subspace of R.*

(a) *If $m \in M(V)$ then there exists $f_m \in V$ such that $\mathrm{in}(f_m) = m$. The polynomial f_m is uniquely determined if we further require*

   that the support of f_m intersects $M(V)$ exactly in m and that
   f_m has leading coefficient 1.

(b) *$M(V)$ is a K-basis of $\mathrm{in}(V)$.*

(c) *The set $\{f_m : m \in M(V)\}$ is a K-basis of V.*

(d) *If V has finite dimension, then $\dim(V) = \dim(\mathrm{in}(V))$.*

(e) *Let $a \in \mathbb{N}^n$ be a positive weight vector. Suppose V is a-graded, say $V = \bigoplus_{i=0}^{\infty} V_i$. Then $\mathrm{in}(V) = \bigoplus_{i=0}^{\infty} \mathrm{in}(V_i)$. In particular, V and $\mathrm{in}(V)$ have the same Hilbert function, i.e. $\dim(V_i) = \dim(\mathrm{in}(V)_i)$ for all $i \in \mathbb{N}$.*

(f) *Let $V_1 \subseteq V_2$ be K-subspaces of R. Then $\mathrm{in}(V_1) \subseteq \mathrm{in}(V_2)$ and the (residue classes of the) elements in $M(V_2) \setminus M(V_1)$ form a K-basis of the quotient space $\mathrm{in}(V_2)/\mathrm{in}(V_1)$. Furthermore the set of the (residue classes of the) f_m with $f_m \in V_2$ and $m \in M(V_2) \setminus M(V_1)$ is a K-basis of V_2/V_1 (regardless of the choice of the f_m).*

(g) *The set of the (residue classes of the) elements in $M(R) \setminus M(V)$ is a K-basis of R/V.*

(h) *Let $V_1 \subseteq V_2$ be K-subspaces of R. If $\mathrm{in}(V_1) = \mathrm{in}(V_2)$, then $V_1 = V_2$.*

(i) *Let V be a K-subspace of R and σ, τ monomial orders. If $\mathrm{in}_\tau(V) \subseteq \mathrm{in}_\sigma(V)$, then $\mathrm{in}_\tau(V) = \mathrm{in}_\sigma(V)$.*

Proof. (a) and (b) follow easily from the fact that the monomials form a K-basis of R. For (a) we have to use that descending chains in $M(R)$ terminate.

To prove (c) one notes that the f_m are linearly independent since they have distinct initial monomials. To show that they generate V, we pick any non-zero $f \in V$ and set $m = \mathrm{in}(f)$. Then $m \in M(V)$ and we may subtract from f a suitable scalar multiple of f_m, say $g = f - \lambda f_m$, so that $\mathrm{in}(g) < \mathrm{in}(f)$, unless $g = 0$. Since $g \in V$, we may repeat the procedure with g and go on in the same manner. By Remark 1.2, after a finite number of steps we reach 0, and f is a linear combination of the polynomials f_m collected in the subtraction procedure.

(d) and (e) follow from (b) and (c) after the observation that the element f_m can be taken a-homogeneous if V is a-graded.

The first two assertions in (f) are easy. For the last we note that f_m can be chosen in V_1 if $m \in \mathrm{in}(V_1)$.

The residue classes of the f_m with $m \in M(V_2) \setminus M(V_1)$ are linearly independent modulo V_1 since otherwise there would be a non-trivial linear combination $g = \sum \lambda_m f_m \in V_1$. But then $\mathrm{in}(g) \in \mathrm{in}(V_1)$, a contradiction since $\mathrm{in}(g)$ is one of the monomials m which by assumption do not belong to $M(V_1)$.

To show that the f_m with $m \in M(V_2) \setminus M(V_1)$ generate V_2/V_1 take some non-zero element $f \in V_2$ and set $m = \text{in}(f)$. Subtracting a suitable scalar multiple of f_m from f we obtain a polynomial in V_2 with smaller initial monomial than f (or 0). If $m \in M(V_1)$, then $f_m \in V_1$. Repeating the procedure we reach 0 after finitely many steps. So f can be written as a linear combination of elements of the form f_m with $m \in M(V_2) \setminus M(V_1)$ and elements of V_1, which is exactly what we want.

(g) is a special case of (f) with $V_2 = R$ since in this case we can take $f_m = m$ for all $m \in M(R) \setminus M(V)$.

(h) follows from (f) since $\text{in}(V_1) = \text{in}(V_2)$ implies that the empty set is a basis of V_2/V_1.

Finally, (i) follows from (g) because an inclusion between the two bases $\{m \in M(R) : m \notin M_\tau(V)\}$ and $\{m \in M(R) : m \notin M_\sigma(V)\}$ of the space R/V implies that they are equal. $\qquad\square$

Remark/Definition 1.5. (a) If A is a K-subalgebra of R, then $\text{in}(A)$ is also a K-subalgebra of R. This follows from equation (1) and from 1.4(a). The K-algebra $\text{in}(A)$ is called the *initial algebra of A* (with respect to τ).

(b) If A is a K-subalgebra of R and J is an ideal of A, then $\text{in}(J)$ is an ideal of the initial algebra $\text{in}(A)$. This, too, follows from equation (1) and from 1.4(a).

(c) If I is an ideal of R, then $\text{in}(I)$ is also an ideal of R. This is a special case of (b) since $\text{in}(R) = R$.

Definition 1.6. Let A be K-subalgebra of R. A subset F of A is said to be a *Sagbi basis* of A (with respect to τ) if the initial algebra $\text{in}(A)$ is equal to the K-algebra generated by the monomials $\text{in}(f)$ with $f \in F$.

If the initial algebra $\text{in}(A)$ is generated, as a K-algebra, by a set of monomials G, then for every m in G we can take a polynomial f_m in A such that $\text{in}(f_m) = m$. Therefore A has a finite Sagbi basis iff $\text{in}(A)$ is finitely generated. However it may happen that A is finitely generated, but $\text{in}(A)$ is not. The following example is given in [15] (with a somewhat different reasoning).

Example 1.7. Let K be an arbitrary field, τ a term order on $K[X,Y]$, and $A = K[X + Y, XY, XY^2]$. The reader may check that A contains all monomials XY^k, $k \geq 1$. Therefore all these monomials belong to $\text{in}(A)$, as well as X if $X > Y$. Now we compute the Hilbert series of A: it is generated by elements of degree $1, 2, 3$ with a relation in degree 6.

So

$$H_A(t) = \frac{1 - t^6}{(1 - t)(1 - t^2)(1 - t^3)} = \frac{1 - t + t^2}{(1 - t)^2}.$$

Then it is easy to check that the Hilbert function takes the value 1 in degree 0 and n in degree n, $n > 0$. The algebra generated by the monomials XY^k, $k \geq 0$, has the same Hilbert function. Since it is contained in the initial algebra (in the case $X > Y$) it is in fact the initial algebra, but certainly not finitely generated.

If $Y > X$, one uses that A is also generated by $X + Y$, YX, and YX^2.

Definition 1.8. Let A be a K-subalgebra of R and J be an ideal of A. A subset F of J is said to be a *Gröbner basis* of J with respect to τ if the initial ideal $\text{in}(J)$ is equal to the ideal of $\text{in}(A)$ generated by the monomials $\text{in}(f)$ with $f \in F$.

If the initial ideal $\text{in}(J)$ is generated, as an ideal of $\text{in}(A)$, by a set of monomials G, then for every m in G we can take a polynomial f_m in J such that $\text{in}(f_m) = m$. Therefore J has a finite Gröbner basis iff $\text{in}(J)$ is finitely generated. In particular, if $\text{in}(A)$ is a finitely generated K-algebra, then it is Noetherian and so all the ideals of A have a finite Gröbner basis. Evidently, all the ideals of R have a finite Gröbner basis.

There is an algorithm to determine a Gröbner basis of an ideal of R starting from any (finite) system of generators, the famous *Buchberger algorithm*. Similarly there is an algorithm that decides whether a given (finite) set of generators for a subalgebra A is a Sagbi basis. There also exists a procedure that completes a system of generators to a Sagbi basis of A, but it does not terminate if the initial algebra is not finitely generated. If a finite Sagbi basis for an algebra A is known, a generalization of Buchberger's algorithm finds Gröbner bases for ideals of A. We refer the interested readers to the literature quoted at the beginning of this section.

2. INITIAL OBJECTS WITH RESPECT TO WEIGHTS

In order to present the deformation theory for initial ideals and algebras we need to further generalize these notions and consider initial objects with respect to weights. As pointed out above, any positive integral weight vector $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$ induces a structure of a positively graded algebra on R. Let t be a new variable and set

$$S = R[t].$$

For $f = \sum \gamma_i m_i \in R$ with $\gamma_i \in K$ and monomials m_i one defines the a-*homogenization* $\hom_a(f)$ of f to be the polynomial

$$\hom_a(f) = \sum \gamma_i m_i t^{a(f)-a(m_i)}.$$

Let $a' = (a_1, \ldots, a_n, 1) \in \mathbb{N}^{n+1}$. Clearly, for every $f \in R$ the element $\hom_a(f) \in S$ is a'-homogeneous, and $f = \hom_a(f)$ iff f is a-homogeneous. One has

$$\begin{aligned} \operatorname{in}_a(fg) &= \operatorname{in}_a(f) \operatorname{in}_a(g) \\ \hom_a(fg) &= \hom_a(f) \hom_a(g) \end{aligned} \qquad \text{for all } f, g \in R. \qquad (2)$$

For every K-subspace V of R we set

$\operatorname{in}_a(V) =$ the K-subspace of R generated by $\operatorname{in}_a(f)$ with $f \in V$,

$\hom_a(V) =$ the $K[t]$-submodule of S generated by $\hom_a(f)$

with $f \in V$.

If A is a K-subalgebra of R and J is an ideal of A, then it follows from (2) that $\operatorname{in}_a(A)$ is a K-subalgebra of R and $\operatorname{in}_a(J)$ is an ideal of $\operatorname{in}_a(A)$. Furthermore $\hom_a(A)$ is a $K[t]$-subalgebra of S and $\hom_a(J)$ is an ideal of $\hom_a(A)$. As for initial objects with respect to monomial orders, $\operatorname{in}_a(A)$ and $\hom_a(A)$ need not be finitely generated K-algebras, even when A is finitely generated. But if $\operatorname{in}_a(A)$ is finitely generated, we may find generators of the form $\operatorname{in}_a(f_1), \ldots, \operatorname{in}_a(f_k)$ with $f_1, \ldots, f_k \in A$. It is easy to see that the f_i generate A. This follows from the next lemma in which we use the notation $f^\alpha = \prod f_i^{\alpha_i}$ for a vector $\alpha \in \mathbb{N}^k$ and the list $f = f_1, \ldots, f_k$.

Lemma 2.1. *Let A be K-subalgebra of R. Assume that $\operatorname{in}_a(A)$ is finitely generated by $\operatorname{in}_a(f_1), \ldots, \operatorname{in}_a(f_k)$ with $f_1, \ldots, f_k \in A$. Then every $F \in A$ has a representation*

$$F = \sum \lambda_i f^{\beta_i}$$

where $\lambda_i \in K \setminus \{0\}$ and $a(F) \geq a(f^{\beta_i})$ for all i.

Proof. By decreasing induction on $a(F)$. The case $a(F) = 0$ being trivial, we assume $a(F) > 0$. Since $F \in A$ we have $\operatorname{in}_a(F) \in \operatorname{in}_a(A) = K[\operatorname{in}_a(f_1), \ldots, \operatorname{in}_a(f_k)]$. Since $\operatorname{in}_a(F)$ is an a-homogeneous element of the a-graded algebra $\operatorname{in}_a(A)$, we may write

$$\operatorname{in}_a(F) = \sum \lambda_i \operatorname{in}_a(f^{\alpha_i})$$

where $a(\operatorname{in}_a(f^{\alpha_i})) = a(\operatorname{in}_a(F))$ for all i. We set $F_1 = F - \sum \lambda_i f^{\alpha_i}$ and conclude by induction since $a(F_1) < a(F)$ if $F_1 \neq 0$. $\qquad \square$

The following lemma contains a simple but crucial fact:

Lemma 2.2. *Let A be a K-subalgebra of R and J be an ideal of A. Assume that $\mathrm{in}_a(A)$ is finitely generated by $\mathrm{in}_a(f_1), \ldots, \mathrm{in}_a(f_k)$ with $f_1, \ldots, f_k \in A$. Let $B = K[Y_1, \ldots, Y_k]$ and take presentations*

$$\varphi_1 : B \to A/J \quad and \quad \varphi : B \to \mathrm{in}_a(A)/\mathrm{in}_a(J)$$

defined by the substitutions $\varphi_1(Y_i) = f_i \bmod (J)$ and $\varphi(Y_i) = \mathrm{in}_a(f_i) \bmod (\mathrm{in}_a(J))$. Set $b = (a(f_1), \ldots, a(f_k)) \in \mathbb{N}_+^k$. Then

$$\mathrm{in}_b(\mathrm{Ker}\,\varphi_1) = \mathrm{Ker}\,\varphi.$$

Proof. As a vector space, $\mathrm{in}_b(\mathrm{Ker}\,\varphi_1)$ is generated by the elements $\mathrm{in}_b(p)$ with $p \in \mathrm{Ker}\,\varphi_1$. Set $u = b(p)$. Then we may write $p = \sum \lambda_i Y^{\alpha_i} + \sum \mu_j Y^{\beta_j}$ where $b(Y^{\alpha_i}) = u$ and $b(Y^{\beta_j}) < u$. The image $F = \sum \lambda_i f^{\alpha_i} + \sum \mu_j f^{\beta_j}$ belongs to J, and, hence, $\mathrm{in}_a(F) \in \mathrm{in}_a(J)$. Since $b(Y^\gamma) = a(f^\gamma)$, it follows that $\mathrm{in}_a(F) = \sum \lambda_i \mathrm{in}_a(f^{\alpha_i})$. Thus $\mathrm{in}_b(p) \in \mathrm{Ker}\,\varphi$, and this proves the inclusion $\subseteq$.

For the other inclusion we lift φ_1 and φ to presentations

$$\rho_1 : B \to A \quad and \quad \rho : B \to \mathrm{in}_a(A),$$

mapping Y_i to f_i and to $\mathrm{in}_a(f_i)$, respectively. Take a system of b-homogeneous generators G_1 of the ideal $\mathrm{Ker}\,\rho$ of B and a system of a-homogeneous generators G_2 of the ideal $\mathrm{in}_a(J)$ of $\mathrm{in}_a(A)$. Every $g \in G_2$, being a-homogeneous of degree $u = a(g)$, is of the form $g = \mathrm{in}_a(g')$, with $g' \in J$. Then $g' = \sum \gamma_i f^{\alpha_i} + \sum \mu_j f^{\beta_j}$ with $a(f^{\alpha_i}) = u$ and $a(f^{\beta_j}) < u$. Therefore $g = \sum \gamma_i \mathrm{in}_a(f^{\alpha_i})$.

We choose the canonical preimage of the given representation of g, i.e. $h_g = \sum \gamma_i Y^{\alpha_i}$. Then the set $G_1 \cup \{h_g : g \in G_2\}$ generates the ideal $\mathrm{Ker}\,\varphi$. For all $g \in G_2$ and g' as above, the canonical preimage of the given representation of g', i.e. $h = \sum \gamma_i Y^{\alpha_i} + \sum \mu_j Y^{\beta_j}$ is in $\mathrm{Ker}\,\varphi_1$, and one has $\mathrm{in}_b(h) = h_g$.

It remains to show that $g \in \mathrm{in}_b(\mathrm{Ker}\,\varphi_1)$ for $g \in G_1$. Every $g \in G_1$ is homogeneous, say of degree u, and hence $g = \sum \lambda_i Y^{\alpha_i}$ with $b(Y^{\alpha_i}) = u$. It follows that $\sum \lambda_i \mathrm{in}_a(f^{\alpha_i}) = 0$. Therefore $\sum \lambda_i f^{\alpha_i} = \sum \mu_j f^{\beta_j}$ with $a(f^{\beta_j}) < u$ by Lemma 2.1. That is, $g' = \sum \lambda_i Y^{\alpha_i} - \sum \mu_j Y^{\beta_j}$ is in $\mathrm{Ker}\,\rho_1$. In particular, $g' \in \mathrm{Ker}\,\varphi_1$ and $\mathrm{in}_b(g') = g$. $\square$

A weight vector a and a monomial order τ on R define a new monomial order τa that "refines" the weight a by τ:

$$m_1 >_{\tau a} m_2 \iff \begin{cases} a(m_1) > a(m_2) \text{ or} \\ a(m_1) = a(m_2) \text{ and } m_1 >_\tau m_2. \end{cases}$$

We extend τa to $S = R[t]$ by setting:

$$m_1 t^i >_{\tau a'} m_2 t^j \iff \begin{cases} a'(m_1 t^i) > a'(m_2 t^j) \text{ or} \\ a'(m_1 t^i) = a'(m_2 t^j) \text{ and } i < j \text{ or} \\ a'(m_1 t^i) = a'(m_2 t^j) \text{ and } i = j \text{ and } m_1 >_\tau m_2. \end{cases}$$

By construction one has

$$\mathrm{in}_{\tau a}(f) = \mathrm{in}_{\tau a'}(\mathrm{hom}_a(f)) \quad \text{for all } f \in R, \ f \neq 0.$$

Given a K-subspace V of R, we let $VK[t]$ denote the $K[t]$-submodule of S generated by the elements in V.

Proposition 2.3. *Let $a \in \mathbb{N}^n$ be a positive integral vector and τ be a monomial order on R. For every K-subspace V of R one has:*

(a) $\mathrm{in}_{\tau a}(V) = \mathrm{in}_{\tau a}(\mathrm{in}_a(V)) = \mathrm{in}_\tau(\mathrm{in}_a(V))$,
(b) *If either* $\mathrm{in}_\tau(V) \subseteq \mathrm{in}_a(V)$ *or* $\mathrm{in}_\tau(V) \supseteq \mathrm{in}_a(V)$, *then* $\mathrm{in}_\tau(V) = \mathrm{in}_a(V)$,
(c) $\mathrm{in}_{\tau a}(V)K[t] = \mathrm{in}_{\tau a'}(\mathrm{hom}_a(V))$,
(d) *The quotient* $S/\mathrm{hom}_a(V)$ *is a free $K[t]$-module.*

Proof. (a) Note that $\mathrm{in}_{\tau a}(f) = \mathrm{in}_{\tau a}(\mathrm{in}_a(f)) = \mathrm{in}_\tau(\mathrm{in}_a(f))$ holds for every $f \in R$. It follows that the first space is contained in the second and in the third. On the other hand, since $\mathrm{in}_a(V)$ is a-homogeneous, the monomials in its initial space are initial monomials of a-homogeneous elements. But every a-homogeneous element in $\mathrm{in}_a(V)$ is of the form $\mathrm{in}_a(f)$ with $f \in V$. This gives the other inclusions.

(b) If one of the two inclusions holds, then an application of $\mathrm{in}_\tau(..)$ to both sides yields that $\mathrm{in}_\tau(V)$ either contains or is contained in $\mathrm{in}_\tau(\mathrm{in}_a(V))$. By (a) the latter is $\mathrm{in}_{\tau a}(V)$. Then by Proposition 1.4(i) we have that $\mathrm{in}_\tau(V) = \mathrm{in}_{\tau a}(V)$. Next we may apply 1.4(h) and conclude that $\mathrm{in}_\tau(V) = \mathrm{in}_a(V)$.

(c) For every $f \in R$ one has $\mathrm{in}_{\tau a'}(\mathrm{hom}_a(f)) = \mathrm{in}_{\tau a}(f)$. Thus $\mathrm{in}_{\tau a}(V)K[t] \subseteq \mathrm{in}_{\tau a'}(\mathrm{hom}_a(V))$. On the other hand, $\mathrm{hom}_a(V)$ is an a'-homogeneous space. Therefore its initial space is generated by the initial monomials of its a'-homogeneous elements. An a'-homogeneous element of degree, say, u in $\mathrm{hom}_a(V)$ has the form $g = \sum_{i=1}^k \lambda_i t^{\alpha_i} \mathrm{hom}_a(f_i)$ where $f_i \in V$ and $\alpha_i + a(f_i) = u$. If $\alpha_i = \alpha_j$ then $a(f_i) = a(f_j)$ and $\mathrm{hom}_a(f_i + f_j) = \mathrm{hom}_a(f_i) + \mathrm{hom}_a(f_j)$. In other words, we may assume that the α_i are all distinct and, after reordering if necessary, that $\alpha_i < \alpha_{i+1}$. Then $\mathrm{in}_{\tau a'}(g) = t^{\alpha_1} \mathrm{in}_{\tau a'}(\mathrm{hom}(f_1)) = t^{\alpha_1} \mathrm{in}_{\tau a}(f_1)$. This proves the other inclusion.

(d) By (c) and Proposition 1.4(b) the (classes of the) elements $t^\alpha m$, $\alpha \in \mathbb{N}$, $m \in M(R) \backslash M(V)$, form a K-basis of $S/\hom_a(V)$. This implies that the set $M(R) \setminus M(V)$ is a $K[t]$-basis of $S/\hom_a(V)$. $\qquad\square$

The next proposition connects the structure of R/I with that of $R/\operatorname{in}_a(R)$:

Proposition 2.4. *For every ideal I of R the ring $S/\hom_a(I)$ is a free $K[t]$-module. In particular $t - \alpha$ is a non-zero divisor on $S/\hom_a(I)$ for every $\alpha \in K$. Furthermore $S/(\hom_a(I) + (t)) \cong R/\operatorname{in}_a(I)$ and $S/(\hom_a(I) + (t - \alpha)) \cong R/I$ for all $\alpha \neq 0$.*

Proof. The first assertion follows from 2.3(d). It implies that every non-zero element of $K[t]$ is a non-zero divisor on $S/\hom_a(I)$. For $S/(\hom_a(I)+(t)) \cong R/\operatorname{in}_a(I)$ it is enough that $\hom_a(I)+(t) = \operatorname{in}_a(I)+(t)$. This is easily seen since for every $f \in R$ the polynomials $\operatorname{in}_a(f)$ and $\hom_a(f)$ differ only by a multiple of t. To prove that $S/(\hom_a(I)+(t-\alpha)) \cong R/I$ for every $\alpha \neq 0$, we consider the graded isomorphism $\psi : R \to R$ induced by $\psi(X_i) = \alpha^{-a_i}X_i$. One checks that $\psi(m) = \alpha^{-a(m)}m$ for every monomial m of R and that $\hom_a(f) - \alpha^{a(f)}\psi(f)$ is a multiple of $t - \alpha$ for all the $f \in R$. So $\hom_a(I) + (t-\alpha) = \psi(I) + (t-\alpha)$, which implies the desired isomorphism. $\qquad\square$

3. The transfer of arithmetic and homological properties

Now we use Proposition 2.4 for comparing R/I with $R/\operatorname{in}_a(I)$.

Proposition 3.1.   (a) *R/I and $R/\operatorname{in}_a(I)$ have the same Krull dimension.*

  (b) *The following properties are passed from $R/\operatorname{in}_a(I)$ on to R/I: being reduced, a domain, a normal domain, Cohen-Macaulay, Gorenstein.*

  (c) *Suppose that I is graded with respect to some positive weight vector b. Then $\operatorname{in}_a(I)$ is b-graded, too, and the Hilbert functions of R/I and $R/\operatorname{in}_a(I)$ coincide.*

Proof. Let us start with (b). The bridge between R/I and $R/\operatorname{in}(I)$ is formed by $A = S/\hom_a(I)$. We have representations $R/I = A/(t-1)$ and $R/\operatorname{in}(I) = A/(t)$. So we must show that the properties under consideration first *ascend* from the residue class ring $A/(t)$ to A and then *descend* from A to $A/(t-1)$.

Ascent. The K-algebra A is positively graded. Let $\mathfrak{m}$ denote its maximal ideal generated by the residue classes of the indeterminates. Set

$A' = A_{\mathfrak{m}}$. Then we have the following commutative diagram in which all maps are the natural ones:

$$
\begin{array}{ccc}
A & \longrightarrow & A' \\
\downarrow & & \downarrow \\
A/(t) & \longrightarrow & A'/(t).
\end{array}
$$

(i) We start from $A/(t)$. The passage to its localization $A'/(t)$ with respect to the maximal ideal $\mathfrak{m}/(t)$ preserves all the properties under consideration.

(ii) Now we have to go up from $A'/(t)$ to A' itself. It is elementary to show that A' is reduced or an integral domain if $A'/(t)$ has this property (see [5, Proof of 2.2.3] for the prototype of such an argument). Normality is covered by the next lemma. For the Cohen-Macaulay and Gorenstein property the conclusion is contained in [5, 2.1.3 and 3.1.9].

(iii) Finally, one observes that A has one of the properties mentioned if and only if its localization $A' = A_{\mathfrak{m}}$ does so. In fact, all of the properties depend only on the localizations of A with respect to graded prime ideals, and such localizations are localizations of A' (see [5, Section 1.5 and Chapters 2 and 3], in particular [5, 2.1.27, 3.6.20]). For the only non-local property, namely that of being an integral domain, one notes that $\mathfrak{m}$ contains all the associated prime ideals of A.

Descent. It remains to transfer the properties in (b) to $A'' = A/(t-1)$ $\cong R/I$. At this point one should observe that A'' is not merely a residue class ring modulo a non-zero-divisor, but in fact the dehomogenization of A with respect to the degree 1 element t. So A'' is the degree 0 component of the graded ring $A[t^{-1}]$, and $A[t^{-1}]$ is just the Laurent polynomial ring in the variable t over A''. (This is not hard to see; cf. [5, Section 1.5]. The main point is that the surjection $A \to A''$ factors through $A[t^{-1}]$ and that the latter ring has a homogeneous unit of degree 1.) Finally, each of the properties descends from the Laurent polynomial ring to A''. The proof of (b) is complete.

For (a) one follows the same chain of descents and ascents:

$$
\dim R/I = \dim A'' = \dim A''[t, t^{-1}] - 1 = \dim A[t^{-1}] - 1
$$
$$
= \dim A - 1 = \dim R/\operatorname{in}(A).
$$

For the equation $\dim A = \dim A[t^{-1}]$ one has to use that t is a non-zero-divisor in the affine K-algebra A: it can not be contained in all maximal ideals $\mathfrak{n}$ of A for which $\dim A = \dim A_{\mathfrak{n}}$. In fact, let $\mathfrak{p}$ be a minimal prime ideal of A with $\dim A = \dim A/\mathfrak{p}$. Then all maximal ideals $\mathfrak{n} \supset \mathfrak{p}$ have $\dim A_{\mathfrak{n}} = \dim A$, and $\mathfrak{p}$ is their intersection. But

$t \notin \mathfrak{p}$. For the very last equation one can use that t is a homogeneous non-zero-divisor in the positively graded ring A.

For (c) one first notes that $\mathrm{in}_a(I)$ is b-graded, since the initial form of a b-homogeneous element is b-homogeneous, too. We refine the weight a by a monomial order τ and derive the chain of equations

$$H(R/\mathrm{in}_a(I)) = H(R/\mathrm{in}_\tau(\mathrm{in}_a(I))) = H(R/\mathrm{in}_{\tau a}(I)) = H(R/I)$$

for the Hilbert function $H(..)$ from 1.4(e) and 2.3(a). $\square$

We have to add a lemma already used in the proof above. The local rings used there are catenary as is every localization of an affine K-algebra.

Lemma 3.2. *Let A be a catenary noetherian local ring and t a non-zero-divisor of A. If $A/(t)$ is normal, then so is A.*

Proof. We must show that A has the Serre properties (R_1) and (S_2) if these hold for $A/(t)$. Let $\mathfrak{p}$ be a prime ideal of A with height $\mathfrak{p} \leq 1$. If $t \in \mathfrak{p}$, then $\overline{\mathfrak{p}} = \mathfrak{p}/(t)$ is a minimal prime ideal of $A/(t)$, and the regularity of $(A/(t))_{\overline{\mathfrak{p}}} = A_{\mathfrak{p}}/(t)$ implies that of $A_{\mathfrak{p}}$. If $t \notin \mathfrak{p}$, we choose a minimal prime overideal $\mathfrak{q}$ of $\mathfrak{p} + (t)$. Since A is catenary, we must have height $\mathfrak{q} =$ height $\mathfrak{p}+1$. Moreover, height $\mathfrak{q}/(t) =$ height $\mathfrak{q}-1 =$ height $\mathfrak{p}$. It follows that $(A/(t))_{\overline{\mathfrak{q}}}$ is regular. So $A_{\mathfrak{q}}$ and its localization $A_{\mathfrak{p}}$ are regular.

Suppose now that height $\mathfrak{p} \geq 2$. We must show that depth$A_{\mathfrak{p}} \geq 2$. If $t \in \mathfrak{p}$, then we certainly have depth$(A/(t))_{\overline{\mathfrak{p}}} \geq 1$, since $(A/(t))_{\overline{\mathfrak{p}}}$ is regular or has depth at least 2. Otherwise we take $\mathfrak{q}$ as above. Then depth$(A/(t))_{\overline{\mathfrak{q}}} \geq 2$, and depth$A_{\mathfrak{q}} \geq 3$. We choose $u \neq 0$ in $\mathfrak{p}$. If depth$A_{\mathfrak{p}} = 1$, then $\mathfrak{p}/(u)$ is an associated prime ideal of $A/(u)$. Moreover, we have depth$A_{\mathfrak{q}}/(u) \geq 2$, and dim $A_{\mathfrak{q}}/\mathfrak{p}A_{\mathfrak{q}} = 1$. This is a contradiction to [5, 1.2.13]: for a local ring R one has depth$R \leq$ dim $R/\mathfrak{p}$ for all associated prime ideals $\mathfrak{p}$ of R. $\square$

Very often one wants to compare finer invariants of $R/\mathrm{in}_a(I)$ and R/I, for example if I is a graded ideal of R with respect to some other weight vector b. The next proposition shows that the comparison is possible for graded components of Tor-modules. The vector space dimensions in the proposition are called *graded Betti numbers*.

Proposition 3.3. *Let a, b positive integral vectors and let J, J_1, J_2 be b-homogeneous ideals of R with $J \subseteq J_1$ and $J \subseteq J_2$. Then $\mathrm{in}_a(J), \mathrm{in}_a(J_1), \mathrm{in}_a(J_2)$ are also b-homogeneous ideals, and one has*

$$\dim_K \mathrm{Tor}_i^{R/J}(R/J_1, R/J_2)_j \leq \dim_K \mathrm{Tor}_i^{R/\mathrm{in}_a(J)}(R/\mathrm{in}_a(J_1), R/\mathrm{in}_a(J_2))_j$$

where the graded structure on the Tor-*modules is inherited from the b-graded structure of their arguments.*

Proof. On S we introduce a bigraded structure, setting $\deg X_i = (b_i, a_i)$ and $\deg t = (0, 1)$. The ideals $I = \hom_a(J)$, $I_1 = \hom_a(J_1)$ and $I_2 = \hom_a(J_2)$ are then bigraded and so are the algebras they define. We need a standard result in homological algebra: if A is a ring, M, N are A-modules and x is a non-zero-divisor on A as well as on M then $\mathrm{Tor}_i^A(M, N/xN) \cong \mathrm{Tor}_i^{A/xA}(M/xM, N/xN)$. (It is difficult to find an explicit reference; for example, one can use [5, 1.1.5].) If, in addition, x is a non-zero-divisor also on N, then we have the short exact sequence $0 \to N \to N \to N/xN \to 0$. It yields the exact sequence

$$0 \to \mathrm{CoKer}\, \varphi_i \to \mathrm{Tor}_i^{A/xA}(M/xM, N/xN) \to \mathrm{Ker}\, \varphi_{i-1} \to 0$$

where φ_i is multiplication by x on $\mathrm{Tor}_i^A(M, N)$.

Set $A = S/\hom_a(J)$, $M = S/\hom_a(J_1)$, $N = S/\hom_a(J_2)$ and $T_i = \mathrm{Tor}_i^A(M, N)$. Since the modules involved are bigraded, so is T_i. Let T_{ij} be the direct sum of all the components of T_i of bidegree (j, k) as k varies. Since T_i is a finitely generated bigraded S-module, T_{ij} is a finitely generated and graded $K[t]$-module (with respect to the standard grading of $K[t]$). So we may decompose it as

$$T_{ij} = F_{ij} \oplus G_{ij}$$

where F_{ij} is the free part and G_{ij} is the torsion part, which, being $K[t]$-graded, is a direct sum of modules of the form $K[t]/(t^a)$ for various $a > 0$. Denote the minimal number of generators of F_{ij} and G_{ij} as $K[t]$-modules by f_{ij} and g_{ij}, respectively. Now we consider the b-homogeneous component of degree j of the above short exact sequence with $x = t$, which is a non-zero-divisor by Proposition 2.3(d). It follows that

$$\dim_K \mathrm{Tor}_i^{R/\mathrm{in}_a(J)}(R/\mathrm{in}_a(J_1), R/\mathrm{in}_a(J_2))_j = f_{ij} + g_{ij} + g_{i-1,j}.$$

If we take $x = t - 1$ instead of x, then we have

$$\dim_K \mathrm{Tor}_i^{R/J}(R/J_1, R/J_2)_j = f_{ij}$$

and this shows the desired inequality. $\square$

Remark 3.4. One can prove an analogous inequality for Ext-modules. However, some care is advisable: the homological degree i changes to $i - 1$ when one passes from A to the residue class rings modulo t and $t - 1$ (Lemma of Rees [5, 3.1.16]).

Note that one can use Proposition 3.3 to transfer the Cohen-Macaulay and Gorenstein properties from $R/\operatorname{in}_a(I)$ to R/I if I is b-graded. We content ourselves with a comparison of two important invariants:

Corollary 3.5. *Under the hypotheses of 3.3 one has*

$$\operatorname{projdim}_R R/I \leq \operatorname{projdim}_R R/\operatorname{in}_a(I).$$

If $(a = (1, \ldots, 1))$, then

$$\operatorname{reg}_R R/I \leq \operatorname{reg}_R R/\operatorname{in}(I).$$

Proof. For both invariants this is an immediate consequence of the proposition, for the projective dimension

$$\operatorname{projdim}_R R/I = \max\{i : \operatorname{Tor}_i^R(R/I, K) \neq 0\}$$

as well as for the Castelnuovo-Mumford regularity

$$\operatorname{reg}_R R/I = \max\{j - i : \operatorname{Tor}_i^R(R/I, K)_j \neq 0\}.$$

(In its definition one assumes that all indeterminates have degree 1.) $\qquad \square$

Remark 3.6. As we will see in Proposition 3.8 every monomial order τ can be approximated by a weight vector a, as long as one only wants to compute the initial ideals of finitely many ideals. Therefore Corollary 3.5 applies also to initial ideals defined by monomial orders.

While the inequalities in the previous corollary are strict in general, they turn into equalities in an important special case, namely when τ is the RevLex order, and the initial ideal is formed after a generic linear transformation γ of the coordinates. Then $\operatorname{gin}(I) = \operatorname{in}_\tau(\gamma(I))$ is called the *generic initial ideal*. One has $\operatorname{projdim}_R R/I = \operatorname{projdim}_R R/\operatorname{gin}(I)$ and $\operatorname{reg}_R R/I = \operatorname{reg}_R R/\operatorname{gin}(I)$; see [8, 19.11 and 20.21] for this theorem of Bayer and Stillman. For further results comparing single Betti numbers of R/I and $R/\operatorname{gin}(I)$ see Bayer, Charalambous and Popescu [3] and Aramova, Herzog and Hibi [1].

If I is graded with respect to the ordinary weight $(1, \ldots, 1)$ then it makes sense to ask for the Koszul property of R/I. By definition, R/I is Koszul if $\operatorname{Tor}_i^{R/I}(R/\mathfrak{m}, R/\mathfrak{m})_j$ is non-zero only for $i = j$. Backelin and Fröberg [2] give a detailed discussion of this class of rings.

Corollary 3.7. *Suppose that I is a graded ideal with respect to the weight $(1, \ldots, 1)$.*

   (a) *If $R/\operatorname{in}_a(I)$ is Koszul for some positive weight a, then R/I is Koszul.*
   (b) *In particular, if $\operatorname{in}_a(I)$ is generated by degree 2 monomials, then R/I is Koszul.*

Proof. (a) follows directly from Proposition 3.3. For (b) one uses a theorem of Fröberg [11]: if J is an ideal generated by quadratic monomials, then the algebra R/J is Koszul. $\qquad\square$

In order to apply the previous results to initial objects defined by monomial orders we have to approximate such orders by weight vectors. This is indeed possible, provided only finitely many monomials have to be considered.

Proposition 3.8. *Let τ be a monomial order on R.*
  (a) *Let $\{(m_1, n_1), \ldots, (m_k, n_k)\}$ be a finite set of pairs of monomials such that $m_i >_\tau n_i$ for all i. Then there exists a positive integral weight $a \in \mathbb{N}_+^n$ such that $a(m_i) > a(n_i)$ for all i.*
  (b) *Let A be a K-subalgebra of R and $I_1, \ldots, I_h$ be ideals of A. Assume that $\operatorname{in}_\tau(A)$ is finitely generated as a K-algebra. Then there exists a positive integral weight $a \in \mathbb{N}_+^n$ such that $\operatorname{in}_\tau(A) = \operatorname{in}_a(A)$ and $\operatorname{in}_\tau(I_i) = \operatorname{in}_a(I_i)$ for all $i = 1, \ldots, h$.*

Proof. (a) Set $m_i = X^{\alpha_i}$ and $n_i = X^{\beta_i}$ and $\gamma_i = \alpha_i - \beta_i \in \mathbb{Z}^n$. Let Γ be the $k \times n$ integral matrix whose rows are the vectors γ_i. We are looking for a positive column vector a such that the coefficients of the vector Γa are all > 0. Suppose, by contradiction, there is no such a. Then (one version of the famous) Farkas Lemma (see Schrijver [16, Section 7.3]) says that there exists a linear combination $v = \sum c_i \gamma_i$ with non-negative integral coefficients $c_i \in \mathbb{N}$ such that $v \leq 0$, that is $v = (v_1, \ldots, v_n)$ with $v_i \leq 0$. Then it follows that $\prod_i m_i^{c_i} X^{-v} = \prod_i n_i^{c_i}$, which contradicts our assumptions because the monomial order is compatible with the semigroup structure.

(b) Let F_0 be a finite Sagbi basis of A, let F_i be a finite Gröbner basis of I_i and set $F = \bigcup_i F_i$. Consider the set U of pairs of monomials $(\operatorname{in}(f), m)$ where $f \in F$ and m is any non-initial monomial of f. Since U is finite, by (a) there exists $a \in \mathbb{N}_+^n$ such that $\operatorname{in}_a(f) = \operatorname{in}_\tau(f)$ for every $f \in H$. We show a has the desired property. Set $V_0 = A$ and $V_i = I_i$. By construction the (algebra for $i = 0$ and ideal for $i > 0$) generators of the $\operatorname{in}_\tau(V_i)$ belong to $\operatorname{in}_a(V_i)$ so that $\operatorname{in}_\tau(V_i) \subseteq \operatorname{in}_a(V_i)$. But then, by Proposition 2.3(b), we may conclude that $\operatorname{in}_\tau(V_i) = \operatorname{in}_a(V_i)$. $\qquad\square$

The main theorem of this section summarizes what we can say about the transfer of ring-theoretic properties from initial objects. For the Koszul property of subalgebras we must allow a "normalization" of degree. Suppose that b is a positive weight vector b, and suppose that a subalgebra A is generated by elements $f_1, \ldots, f_s$ of the same b-degree $e \in \mathbb{N}$. Then every element g of A has b-degree divisible by e, and dividing the b-degree by e we obtain the *e-normalized b-degree* of g.

Theorem 3.9. *Let* $\mathrm{in}(..)$ *denote the initial objects with respect to a positive integral vector* $a \in \mathbb{N}^n$ *or to a monomial order* τ *on* R. *Let* A *be a* K-*subalgebra of* R *and* J *be an ideal of* A. *Suppose that* $\mathrm{in}(A)$ *is finitely generated.*

(a) *One has* $\dim A/J = \dim \mathrm{in}(A)/\mathrm{in}(J)$.
(b) *If* $\mathrm{in}(A)/\mathrm{in}(J)$ *is reduced, a domain, a normal domain, Cohen-Macaulay, or Gorenstein, then so is* A/J.
(c) *Let b be a positive weight vector, and suppose that* A *and* J *are* b-*graded. Then* A/J *and* $\mathrm{in}(A)/\mathrm{in}(J)$ *have the same Hilbert function.*
(d) *If, in addition to the hypothesis of (c),* $\mathrm{in}(A)/\mathrm{in}(J)$ *is Koszul with respect to* e-*normalized* b-*degree for some* e, *then so is* A/J.

Proof. If the initial objects are formed with respect to a monomial order then, by 3.8, we may represent them as initial objects with respect to a suitable positive integral weight vector. Therefore in both cases the initial objects are taken with respect to a positive integral weight a. By Lemma 2.2 there exist a polynomial ring, say B, an ideal H, and a positive weight c such that $B/H \cong A/J$ and $B/\mathrm{in}_c(H) \cong \mathrm{in}(A)/\mathrm{in}(J)$. Furthermore, under the hypothesis of (c), the weight b can be lifted from the generators of $\mathrm{in}(A)$ to the indeterminates of B. Now the theorem follows from Proposition 3.1 and Lemma 3.7. □

The theorem is usually applied in two extreme cases. In the first case $A = R$, so that $\mathrm{in}(A) = R$, and in the second case $H = 0$, so that $\mathrm{in}(J) = 0$.

There is a special instance of the theorem that deserves a separate statement.

Corollary 3.10. *Let* A *be* K-*subalgebra of* R, *and suppose that* $\mathrm{in}(A)$ *is generated by finitely many monomials (e.g. if it is finitely generated and the initial algebra is taken with respect to a monomial order). If* $\mathrm{in}(A)/\mathrm{in}(I)$ *is normal, then* A/I *is normal and Cohen-Macaulay.*

Proof. The hypothesis implies that $\mathrm{in}(I)$ is a prime ideal in the affine semigroup ring $\mathrm{in}(A)$. But then the natural homomorphism $\mathrm{in}(A) \to \mathrm{in}(A)/\mathrm{in}(I)$ splits as a ring homomorphism (see [5, Section 6.1]). It follows that $\mathrm{in}(A)/\mathrm{in}(I)$ is itself a normal affine semigroup ring. By a theorem of Hochster [5, 6.3.5] such a ring is Cohen-Macaulay. □

Sometimes one of the implications in Theorem 3.9 can be reversed:

Corollary 3.11. *Let b be a positive weight vector, and suppose that the* K-*subalgebra* A *is* b-*graded and has a Cohen-Macaulay initial algebra* $\mathrm{in}(A)$. *Then* A *is Gorenstein iff* $\mathrm{in}(A)$ *is Gorenstein.*

Proof. Since in(A) is Cohen-Macaulay, A is Cohen-Macaulay as well. So both algebras are positively graded Cohen-Macaulay domains. By a theorem of Stanley [5, 4.4.6], the Gorenstein property of such rings depends only on their Hilbert function, and both algebras have the same Hilbert function.   □

We want to extend Theorem 3.9 in such a way that it allows us to determine the canonical module of A/I. First a lemma that covers the most difficult step in the passage from in(A)/in(I) to A/I.

Lemma 3.12. *Let R be a positively graded algebra over a field K and C a finitely generated graded R-module. Suppose that $t \in R$ is a homogeneous non-zero-divisor for both R and C. Then C is the canonical module of R (up to the shift a) if (and only if) C/tC is the canonical module of $R/(t)$ (up the shift $a + \deg t$).*

Proof. Let $d = \dim R$. Then $\dim R/(t) = d - 1$. We can assume that $a = 0$, shifting C and C/tC by $-a$ if necessary. By the lemma of Rees (for example, see [5, 3.1.16 and 4.2.40]) we have

$$\mathrm{Ext}^i_{R/(t)}\big(K, (C/tC)(\deg t)\big) \cong \mathrm{Ext}^{i+1}_R(K, C) = \begin{cases} 0 & i \neq d - 1, \\ K & i = d - 1. \end{cases}$$

(with K in degree 0). This property is exactly the definition of the graded canonical module; see [5, Section 3.6].   □

Theorem 3.13. *Let A be a subalgebra of R as in Theorem 3.9, and $I \subseteq J$ ideals of A. Suppose that $\mathrm{in}(A)/\mathrm{in}(I)$ and, hence, A/I are Cohen-Macaulay.*

(a) *If $\mathrm{in}(J)/\mathrm{in}(I)$ is the canonical module of $\mathrm{in}(A)/\mathrm{in}(I)$, then J/I is the canonical module of A/I.*
(b) *Suppose in addition that A, I, J are b-graded with respect to a positive weight and $\mathrm{in}(J)/\mathrm{in}(I)$ is the canonical module of $\mathrm{in}(A)/\mathrm{in}(I)$ (up to a shift). Then J/I is the graded canonical module (up to the same shift).*

Proof. (a) As in the proof of Theorem 3.9 we may assume that the initial objects are defined by a weight vector. Then we choose representations $A/I \cong B/I_1$, $A/J \cong B/I_2$, $\mathrm{in}(A)/\mathrm{in}(I) \cong B/\mathrm{in}(I_1)$, $\mathrm{in}(A)/\mathrm{in}(J) \cong B/\mathrm{in}(I_2)$ as in Lemma 2.2. This reduces the problem to the situation of Proposition 3.1: R is a polynomial ring over K, $I \subset J$ are ideals, and $R/\mathrm{in}(I)$ is Cohen-Macaulay with canonical module $\mathrm{in}(J)/\mathrm{in}(I)$.

Again one passes to the homogenized objects in $S = R[t]$. Note that t and $t-1$ are non-zero-divisors modulo hom(I) and hom(J). Set

$\bar{S} = S/\hom(I)$ and $\bar{J} = \hom(J)/\hom(I)$. Then $\bar{S}/t\bar{S} \cong R/\operatorname{in}(I)$ and

$$\bar{J}/t\bar{J} \cong (\bar{J} + t\bar{S})/(t\bar{S}) \cong \operatorname{in}(J)/\operatorname{in}(I).$$

By Lemma 3.12 we therefore conclude that $\bar{J}$ is the canonical module of $\bar{S}$. But we also have $\bar{J}/(t-1)\bar{J} \cong (\bar{J} + (t-1)\bar{S})/(t-1)\bar{S} \cong J/I$. This shows that J/I is the canonical module of R/I.

(b) It only remains to control the shift. This can be done via the Hilbert functions of A/I and J/I on the one side and those of $\operatorname{in}(A)/\operatorname{in}(I)$ and $\operatorname{in}(J)/\operatorname{in}(I)$ on the other (see [5, 4.4.5]). But the Hilbert functions of the objects corresponding to each other via in coincide, and the claim follows. □

Remark 3.14. In addition to Cohen-Macaulay and Gorenstein rings one can also consider those with rational singularities (in characteristic 0) or F-rational singularities (in characteristic p). They behave well under the deformation to the initial objects. See [7] for a more detailed discussion.

In particular A/I is (F-)rational under the hypotheses of Corollary 3.10.

Remark 3.15. All the results above suggest that the numerical invariants and structural properties can only improve in the direction from $B' = \operatorname{in}(A)/\operatorname{in}(I)$ to $B = A/I$. However, some caution is advisable.

(a) If both algebras are normal Noetherian domains, then one can consider their divisor class groups $\operatorname{Cl}(B)$ and $\operatorname{Cl}(B')$. A potential theorem comparing $\operatorname{Cl}(B)$ and $\operatorname{Cl}(B')$ could be that $\operatorname{Cl}(B)$ is always of the form G/H where $H \subset G \subset \operatorname{Cl}(B')$. This is not the case as the following example indicates.

Choose $A = \mathbb{C}[X^2 - Z^2, XY, Y^2, YZ] \cong \mathbb{C}[T.U.V.W]/(U^2 - TV - W^2)$, its initial algebra with respect to Lex is $\operatorname{in}(A) = \mathbb{C}[X^2, XY, Y^2, YZ] \cong \mathbb{C}[T, U, V, W]/(U^2 - TV)$. For the verification of the claims in this statement it is enough to observe that $\mathbb{C}[X^2, XY, Y^2, YZ]$ has indeed the representation given, and that $U^2 - TV - W^2$ is a relation of the generators of A. The rest follows from Hilbert function arguments.

According to Fossum [10, 11.4] A has divisor class group isomorphic to $\mathbb{Z}$ since the quadratic form defining it is non-degenerate (and $\mathbb{C}$ is algebraically closed). However, $\operatorname{in}(A)$ has class group $\mathbb{Z}/2\mathbb{Z}$.

(b) Another (and related invariant) is the Grothendieck group $K_0(R)$. Gubeladze [13] has given an example of an algebra A with $K_0(A) \neq 0$ for which $\operatorname{in}(A)$ has trivial K_0.

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