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Parametrized Family of 2-D Non-factorable FIR Lossless Systems and Gröbner Bases

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Abstract. The factorability of one-dimensional (1-D) FIR lossless transfer matrices [1] in terms of Givens rotations produces the parameters that can be used for an optimal design of filter banks with prespecified filtering characteristics. Two dimensional (2-D) FIR lossless systems behave quite differently, however. Venkataraman-Levy [2] and Basu-Choi-Chiang [3] have constructed 2-D FIR paraunitary matrices of McMillan degrees $(2,2)$ that are not factorable. Because of the state-space realization used in the construction, they are floating-point approximations, and they do not produce explicit parametrizations that can be used for optimal design process. In this paper, we formulate the lossless condition and nonfactorability condition of a 2-D FIR paraunitary matrix using multivariate polynomials in the coefficients. The resulting polynomial system can be explicitly solved with Gröbner bases. By studying the polynomial system, we obtain a continuous one parameter family of 2-D 2×2 non-factorable paraunitary matrices. As an example, we get a closed-form expression for a 2-D 2×2 paraunitary matrix that is not factorable into rotations and delays.

Key Words: losses, paraunitary, Gröbner bases, non-factorable, parametrization

1. Introduction

One-dimensional (1-D) FIR lossless transfer matrices have been well studied in the past, and are known to demonstrate nice properties. Especially, their factorability in terms of Givens rotations produces parameters that can be used for an optimal design of filter banks with prespecified filtering characteristics [1]. Two dimensional (2-D) FIR lossless systems behave quite differently, however. Venkataraman-Levy [2] and Basu-Choi-Chiang [3] have constructed 2-D FIR paraunitary matrices of McMillan degrees $(2,2)$ that are not factorable. Both examples were constructed using the state-space realization, and thus are expressed as floating-point approximations. Also, they do not produce parametrizations that can be used for optimal design process.

Definition 1.1. 1. For a square Laurent polynomial matrix $\mathbf{H}(z_1,\ldots,z_m)$, its parahermitian conjugate $\tilde{\mathbf{H}}(z_1,\ldots,z_m)$ is

$$\tilde{\mathbf{H}}(z_1,\ldots,z_m) := \mathbf{H}(z_1^{-1},\ldots,z_m^{-1})^t.$$

2. $\mathbf{H}(z_1,\ldots,z_m)$ is called **paraunitary** *if it satisfies* $\tilde{\mathbf{H}}\mathbf{H}=\mathbf{H}\cdot\tilde{\mathbf{H}}=I$.

Recall that an m-dimensional FIR system is represented by a transfer matrix $\mathbf{H}(z_1,\ldots,z_m)$ whose entries are Laurent polynomials. This transfer matrix is paraunitary if the underlying system is a lossless system. Since the inverse of a paraunitary matrix is also paraunitary and belongs to the ring of Laurent polynomials, the associated filter bank has both synthesis and analysis filters that are FIR and satisfy the perfect reconstruction property.

Remark 1.2. It is assumed in this paper that the field of coefficients is $\mathbb{R}$. Most results of this paper can be readily extended to the case of coefficient field $\mathbb{C}$. Over $\mathbb{C}$, the transpose should be replaced by conjugate transpose.

When $m = 0$, paraunitary matrices are just ordinary orthogonal matrices and, are simply products of rotations (up to sign).

When $m = 1$ (univariate case), a classification theorem on the paraunitary matrices was obtained by P. P. Vaidyanathan [1], which asserts that the group of paraunitary matrices is generated by *rotations* (constant unitary matrices) and *delays* (diagonal matrices with monomial entries).

This result can not be extended to the multivariate case (i.e. $m > 1$) as demonstrated by counter-examples mentioned above.

In this paper, we formulate the lossless condition and non-factorability condition of a 2-D FIR paraunitary matrix using multivariate polynomials in the coefficients. The resulting polynomial system can be explicitly solved with Gröbner bases. By taking a convex geometric approach with the polynomial system, we obtain an explicit parametrization of 2-D 2×2 non-factorable paraunitary matrices. By specializing at specific values of the parameters, we get a closed-form expression for a 2-D 2×2 paraunitary matrix that is not factorable into rotations and delays.

It is noted that this type of parametrization is quite relevant in the field of 2-D nonseparable FIR filter bank design, lossless FIR filter bank realization and other related areas. For example, J. Kovačević and M. Vetterli [4] used the parametrization of 2-D factorable FIR paraunitary matrices (in terms of rotations and delays) to construct a nonfactorable 2-D wavelet of high regularity.

Remark 1.3. The approach taken in this paper can be extended to the higher dimensions ($m > 2$) in a straightforward manner.

NOTATION: *If* $\mathbf{H}(z_1, z_2)$ *is a transfer matrix representing a causal system, then all of its entries are polynomials in* z_1^{-1} *and* z_2^{-1}*, i.e. no monomials involved have positive powers. For the notational and computational convenience, we will replace* z_1^{-1} *by* x *and* z_2^{-1} *by* y*. In this convention, a polynomial matrix* $\mathbf{H}(x, y)$ *represents a causal system.*

2. The Structure of 2-D Paraunitary Matrices

LEMMA 2.1: *The determinant of a paraunitary matrix* $\mathbf{H}(x, y)$ *is a monomial of the form* $\pm x^{n_1} y^{n_2}$ *for some* $n_1, n_2 \in \mathbb{Z}$.

Proof: From $\mathbf{H}\cdot\tilde{\mathbf{H}}=\boldsymbol{I}$, we see that $\det(\mathbf{H})\cdot\det(\tilde{\mathbf{H}})=1$, i.e. $\det(\mathbf{H})$ is an invertible element of the Laurent polynomial ring $\mathbb{R}[x^{\pm 1},y^{\pm}]$, which must be a monomial. Therefore, $\det(\mathbf{H})=\pm x^{n_1}y^{n_2}$. □

Multiplying by a delay if necessary, we assume that all the paraunitary matrices being considered are polynomial matrices.

Proposition 2.2 Let $\mathbf{H}(x,y)$ be a square paraunitary **polynomial** matrix with determinant $\pm x^{n_1}y^{n_2}$. Then it can be written uniquely in the following form:

$$\mathbf{H}(x,y)=\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\mathbf{h}_{ij}x^iy^j$$

where $\mathbf{h}_{ij}$*'s are square matrices with constant entries.*

Proof: Since $\mathbf{H}(x,y)$ can be uniquely written as

$$\mathbf{H}(x,y)=\sum_{(i,j)\in I}\mathbf{h}_{ij}x^iy^j$$

for a finite index set I, it remains to show that $\mathbf{h}_{ij}=0$ if $i>n_1$ or $j>n_2$. Let $\delta_1:=\max\{i|h_{ij}\neq 0 \text{ for some } j\}$ and $\delta_2:=\max\{j|h_{ij}\neq 0 \text{ for some } i\}$.

Claim: $\delta_1=n_1$ and $\delta_2=n_2$.

One can write

$$\mathbf{H}(x,y)=\mathbf{h}_0(y)+\mathbf{h}_1(y)x+\cdots+\mathbf{h}_{\delta_1}(y)x^{\delta_1}$$

for some polynomial matrices $\mathbf{h}_i(y)$ with $\mathbf{h}_{\delta_1}(y)$ being not identically zero. Now, let $e^{i\theta}$ be any fixed point on the unit circle. Then $\mathbf{H}(x,e^{i\theta})$ is a univariate paraunitary matrix whose determinant has degree n_1. By using the Vaidyathan-type factorization into rotations and delays, one sees that the highest degree term in the expansion of $\mathbf{H}(x,e^{i\theta})$ with respect to x is precisely x^{n_1}. This means δ_1 is at least n_1. Suppose that $\delta_1>n_1$. Then, $\mathbf{h}_{\delta_1}(e^{i\theta})=0$. Now that this is true for any point $e^{i\theta}$ on the unit circle, by analytic continuation, the polynomial matrix $\mathbf{h}_{\delta_1}(y)$ must be a zero matrix. This contradicts the definition of δ_1. Therefore, $\delta_1=n_1$. And a similar method gives $\delta_2=n_2$. □

Definition 2.3. For any polynomial matrix $\mathbf{G}\in \mathrm{M}_{lm}(R[x,y])$ with the minimal representation $\mathbf{G}=\Sigma_{i=0}^{k_1}\Sigma_{j=0}^{k_2}\,\mathbf{h}_{ij}\,x^iy^j$, we say $\mathbf{G}$ is **of type** $(k_1,k_2)\in\mathbb{Z}^2$ and we call $k=k_1+k_2$ the **total degree** of $\boldsymbol{G}$.

Using this new terminology, Proposition 2.2 can be rephrased as follows:

COROLLARY 2.4: *The type of a paraunitary polynomial matrix is equal to the exponent vector of its determinant.*

An immediate but very useful corollary of this lemma is,

COROLLARY 2.5: *Let* $\mathbf{H}(x,y)$ *be a square paraunitary polynomial matrix. If the determinant of* $\mathbf{H}(x,y)$ *does not involve the variable* x (y*, resp.*), *then* $\mathbf{H}$ is a polynomial matrix that does not involve the variable x (y*, resp.*).

Let $\mathbf{H}(x,y)$ be a 2×2 paraunitary polynomial matrix, and let $\mathbf{v}(x,y)$ be its first column vector. Then the factorability of $\mathbf{H}(x,y)$ clearly implies the factorability of $\mathbf{v}(x,y)$, that is, if

$$\mathbf{H}(x,y) = \mathbf{H}_1(x,y)\mathbf{H}_2(x,y)$$

for two paraunitary polynomial matrices $\mathbf{H}_1(x,y)$ and $\mathbf{H}_2(x,y)$, then

$$\mathbf{v} = \mathbf{H}_1\mathbf{v}_2$$

where $\mathbf{v}_2$ is the first column vector of $\mathbf{H}_2$.

Now, the following lemma asserts that the converse is also true, thereby relating the factorability of a paraunitary matrix with that of a unit norm vector. Since, from a computational point of view, the unit norm vectors are easier to deal with than the paraunitary matrices, we will actually consider the factorability of the polynomial vectors of unit norm rather than that of the paraunitary matrices.

LEMMA 2.6: *Let* $\mathbf{H}(x,y)$ *be a* 2×2 *paraunitary matrix with* $\det(\mathbf{H}) = x^{n_1}y^{n_2}$, *and* $\mathbf{v}$ *be its first column vector. Suppose* $\mathbf{v}$ *is perfectly factorable, i.e.*

$$\mathrm{v} = \mathbf{U}_d\mathbf{D}_d\mathbf{U}_{d-1}\mathbf{D}_{d-1}\cdots\mathbf{U}_1\mathbf{D}_1\mathrm{v}_0 \tag{2.1}$$

where $\mathbf{U}_1,\ldots,\mathbf{U}_d$ are *2×2 orthogonal matrices,* $\mathbf{v}_0\in\mathbb{R}^2$ *is a constant unit norm vector, and each delay* $\mathbf{D}_i$ *is either* $\begin{pmatrix}1 & 0\\ 0 & x\end{pmatrix}$ or $\begin{pmatrix}1 & 0\\ 0 & y\end{pmatrix}$. *Then* $\mathbf{H}(x,y)$ *is also perfectly factorable.*

Proof: Note that $\Pi_{i=1}^{d}\ |\mathbf{D}_i| = x^{k_1}y^{k_2}$ for some positive integers k_1 and k_2. Write $\mathbf{v}_0=$ $\mathbf{v}_0= \begin{pmatrix}a\\ b\end{pmatrix}$ for $a,b\in\mathbb{R}$ with $a^2+b^2=1$. Then one checks easily that

$$\mathbf{H} = \mathbf{U}_d\mathbf{D}_d\mathbf{U}_{d-1}\mathbf{D}_{d-1}\cdots\mathbf{U}_1\mathbf{D}_1\begin{pmatrix}a & -x^{n_1-k_1}y^{n_2-k_2}b\\ b & x^{n_1-k_1}y^{n_2-k_2}a\end{pmatrix}$$

$$= \mathbf{U}_d\mathbf{D}_d\mathbf{U}_{d-1}\mathbf{D}_{d-1}\cdots\mathbf{U}_1\mathbf{D}_1\begin{pmatrix}a & -b\\ b & a\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & x^{n_1-k_1}y^{n_2-k_2}\end{pmatrix}.$$

□

3. Gröbner Bases and Factorability

Consider a vector $\mathbf{v}=\binom{f}{g}\in(\mathbb{R}[x,y])^2$ of unit norm. Then $\mathbf{v}$ satisfies

$$\begin{aligned}\tilde{\mathrm{v}}\,\mathrm{v} &= (\tilde{f}\ \tilde{g})\begin{pmatrix} f \\ g \end{pmatrix} \\ &= \tilde{f}f + \tilde{g}g \\ &= f(x^{-1},y^{-1})f(x,y) + g(x^{-1},y^{-1})g(x,y) \\ &= 1.\end{aligned}$$

The component polynomials $f,g\in\mathbb{R}[\boldsymbol{x}]$ are constrained by the unit norm condition $\tilde{\mathbf{v}}\mathbf{v}=1$, and this constraint can be described by a system of quadratic polynomials in the coefficients of f, g. Let us determine when these algebraic relations describing the unit norm condition on $\mathbf{v}$ guarantee the decomposition of $\mathbf{v}$ as in Lemma 2.6.

Let $\mathbf{v}=\binom{f}{g}\in(\mathbb{R}[x,y])^2$ be of unit norm of type (k_1,k_2) with total degree d, and let $\alpha=x^{k_1}y^{k_2}\in\mathbb{R}[x,y]$. Then $\alpha\tilde{\mathbf{v}}$ becomes a polynomial vector. Define a paraunitary matrix $\mathbf{H}\in M_2(\mathbb{R}[x,y])$ by

$$\mathbf{H} = \begin{pmatrix} f & -\alpha\tilde{g} \\ g & \alpha\tilde{f} \end{pmatrix}.$$

If the total degree $d=2$, then we get the following three cases to consider; $\det(\mathbf{H})=x^2$, xy, y^2.

In terms of $\mathbf{v}$, we see that $\mathbf{v}$ is of type $(2,0)$, $(1,1)$ and $(0,2)$, in respective cases.

The two cases when $\mathbf{v}$ is of type $(2,0)$ and of type $(0,2)$ are trivial by Corollary 2.5 since these are just univariate cases.

Suppose, therefore, $\mathbf{v}$ is of type $(1,1)$, i.e. $\det(\mathbf{H})=xy$. In this case, by Proposition 2.2, we can write

$$\mathbf{H} = \mathbf{h}_{00} + \mathbf{h}_{10}x + \mathbf{h}_{0,1}y + \mathbf{h}_{11}xy$$

for some constant matrices $\mathbf{h}_{ij}$'s. The corresponding expression for $\mathbf{v}$ is,

$$\mathbf{v} = \mathbf{v}_{00} + \mathbf{v}_{10}x + \mathbf{v}_{0,1}y + \mathbf{v}_{11}xy$$

for some $\mathbf{v}_{00},\mathbf{v}_{10},\mathbf{v}_{0,1},\mathbf{v}_{11}\in\mathbb{R}^2$. Define the real numbers $a_{\rm ij},b_{\rm ij}$, $0\leq i,j\leq 1$, by $\mathbf{v}_{ij}=\begin{pmatrix} a_{ij} \\ b_{ij}\end{pmatrix}$ and consider

$$
\begin{aligned}
1 &= \tilde{\mathbf{v}}\mathbf{v} \\
&= (\mathbf{v}_{00}^t + \mathbf{v}_{10}^t x^{-1} + \mathbf{v}_{0,1}^t y^{-1} + \mathbf{v}_{11}^t x^{-1}y^{-1})(\mathbf{v}_{00} + \mathbf{v}_{10}x + \mathbf{v}_{0,1}y + \mathbf{v}_{11}xy).
\end{aligned}
$$

Equating the respective coefficients of x and y, we get the following set of relations:

$$
\begin{aligned}
0 &= \mathbf{v}_{11}^t\mathbf{v}_{00} = a_{01}\,a_{10} + b_{01}\,b_{10} \\
0 &= \mathbf{v}_{10}^t\mathbf{v}_{01} = a_{00}\,a_{11} + b_{00}\,b_{11} \\
0 &= \mathbf{v}_{10}^t\mathbf{v}_{00} + \mathbf{v}_{11}^t\mathbf{v}_{01} = a_{00}\,a_{10} + a_{01}\,a_{11} + b_{00}\,b_{10} + b_{01}\,b_{11} \\
0 &= \mathbf{v}_{01}^t\mathbf{v}_{00} + \mathbf{v}_{11}^t\mathbf{v}_{10} = a_{00}\,a_{01} + a_{10}\,a_{11} + b_{00}\,b_{01} + b_{10}\,b_{11} \\
1 &= \mathbf{v}_{00}^t\mathbf{v}_{0,0} + \mathbf{v}_{01}^t\mathbf{v}_{0,1} + \mathbf{v}_{10}^t\mathbf{v}_{1,0} + \mathbf{v}_{11}^t\mathbf{v}_{11} \\
&= a_{00}^2 + a_{01}^2 + a_{10}^2 + a_{11}^2 + b_{00}^2 + b_{01}^2 + b_{10}^2 + b_{11}^2.
\end{aligned} \tag{3.1}
$$

Note here that the above set of relations gives defining equations for a unit norm vector of type bounded by (1,1), that is, if we choose any real numbers a_{ij}'s and b_{ij}'s satisfying above set of relations, and define a polynomial vector $\mathbf{v}$ by $\mathbf{v}=\sum_{i=0}^{1}\sum_{j=0}^{1}\begin{pmatrix} a_{ij} \\ bij\end{pmatrix} x^i y^j$, then $\mathbf{v}$ will be a unit norm vector of type $\leq$(1,1).

Define the five polynomials h_i, $1\leq i\leq 5$ by

$$
\begin{aligned}
h_1 &= a_{01}\,a_{10} + b_{01}\,b_{10} \\
h_2 &= a_{00}\,a_{11} + b_{00}\,b_{11} \\
h_3 &= a_{00}\,a_{10} + a_{01}\,a_{11} + b_{00}\,b_{10} + b_{01}\,b_{11} \\
h_4 &= a_{00}\,a_{01} + a_{10}\,a_{11} + b_{00}\,b_{01} + b_{10}\,b_{11} \\
h_5 &= a_{00}^2 + a_{01}^2 + a_{10}^2 + a_{11}^2 + b_{00}^2 + b_{01}^2 + b_{10}^2 + b_{11}^2 - 1.
\end{aligned}
$$

Note that each h_i is an element of the polynomial ring

$$\mathbb{R}[a_{00}, a_{10}, a_{01}, a_{11}, b_{00}, b_{10}, b_{01}, b_{11}].$$

Then, the fixed values of a_{ij}'s and b_{ij}'s satisfying the set of relations in (3.1) can be described as the set $V(h_1,\ h_2,\ h_3,\ h_4,\ h_5)\subset\mathbb{R}^8$, i.e. the set of common zeros of the polynomials h_i's. The set $V(h_1,\ h_2,\ h_3,\ h_4,\ h_5)$ will be called the **Paraunitary Variety of type** (1,1).

The real valued points on this subvariety of affine 8-space are in one-to-one correspondence with the unit norm polynomial vectors of type $\leq$(1,1), and thus paraunitary matrices of determinant $x^{n_1}y^{n_2}$ with $(n_1, n_2)\leq$(1,1). Therefore, this variety precisely parametrizes all the paraunitary matrices whose determinant is a factor of xy.

To see what type of algebraic relations on a_i's and b_i's assure the factorability of $\mathbf{v}$ as in Lemma 2.6, assume

$$\mathbf{v} = \mathbf{R}(\theta)\mathbf{D}(x)\mathbf{v}' \tag{3.2}$$

for a certain rotation matrix $\mathbf{R}(\theta)$, the delay $\mathbf{D}(x) = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}$, and a polynomial vector $\mathbf{v}' \in (\mathbb{R}[x,y])^2$. Letting $\mathbf{v}_0(y) = \mathbf{v}_{00} + \mathbf{v}_{01}y$ and $\mathbf{v}_1(y) = \mathbf{v}_{10} + \mathrm{v}_{11}y$, we get

$$\begin{aligned} \mathbf{R}(\theta)^t\mathbf{v} &= \mathbf{R}(\theta)^t(\mathbf{v}_0(y) + \mathbf{v}_1(y)x) \\ &= \mathbf{R}(\theta)^t\mathbf{v}_0(y) + \mathbf{R}(\theta)^t\mathbf{v}_1(y)x \\ &= \mathbf{D}(x)\mathbf{v}' = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}\mathbf{v}'. \end{aligned}$$

Since the second component of the vector $\mathbf{R}(\theta)^t\mathbf{v}$ is divisible by x, its constant term, that is, the second component of the vector $\mathbf{R}(\theta)^t\mathbf{v}_0(y)$, should be zero. So, we get $(-\sin(\theta),\cos(\theta))\mathbf{v}_0(y) = 0$, i.e.

$$(-\sin(\theta), \cos(\theta))\mathbf{v}_{00} = (-\sin(\theta), \cos(\theta))\mathbf{v}_{01} = 0.$$

Conversely, if there exists a nonzero constant vector (a, b) such that $(a, b)\mathbf{v}_0(y) = 0$, then $\mathbf{v}$ splits as in (3.2), with $\mathbf{R}(\theta) = \frac{1}{\sqrt{a^2+b^2}}\begin{pmatrix} b & a \\ -a & b \end{pmatrix}$. We can do the same to see when $\mathbf{v}$ splits with the factor $\mathbf{R}(\theta)\mathbf{D}(y)$. The following summarizes this observation:

The vector $\mathbf{v}$ splits as in (2.1)

$\iff$ "$(\mathrm{a}, \mathrm{b})\ \mathbf{v}_{00} = (\mathrm{a}, \mathrm{b})\ \mathbf{v}_{01} = 0$" or "$(\mathrm{a}, \mathrm{b})\ \mathbf{v}_{00} = (\mathrm{a}, \mathrm{b})\ \mathbf{v}_{10} = 0$" has a nontrivial solution (a, b).

$\iff \begin{pmatrix} a_{00} & b_{00} \\ a_{01} & b_{01} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \mathbf{0}$ or $\begin{pmatrix} a_{00} & b_{00} \\ a_{10} & b_{10} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \mathbf{0}$ has a

nontrivial solution $\begin{pmatrix} a \\ b \end{pmatrix}$.

$$\iff \mathbf{v}_{00} \parallel \mathbf{v}_{01} \textbf{ or } \mathbf{v}_{00} \parallel \mathbf{v}_{10}. \tag{3.3}$$

$$\iff a_{00}b_{01} - b_{00}a_{01} = 0 \text{ or } a_{00}b_{10} - b_{00}a_{10} = 0.$$

$$\iff h := (a_{00}b_{01} - b_{00}a_{01})(a_{00}b_{10} - b_{00}a_{10}) = 0. \tag{3.4}$$

Let $I(1,1) \subset \mathbb{R}[a_{00}, a_{10}, a_{01}, a_{11}, b_{00}, b_{10}, b_{01}, b_{11}]$ be the ideal generated by the 5 quadratic polynomials h_i, $1 \le i \le 5$. Since any point in the variety $V(I(1,1))$ defines a unit norm vector of type $\le(1,1)$ and an arbitrary point in $V(h)$ defines a vector decomposable as in Lemma 2.6, showing $V(I(1,1)) \subset V(h)$ is equivalent to showing that any unit norm vector of type $\le(1,1)$ is factorable as in Lemma 2.6. Therefore, the problem of determining the factorability of $\mathbf{v}$ boils down to:

$$V(I(1,1)) \quad \subset \quad V(h)? \tag{3.5}$$

This is a well known problem in computational algebraic geometry which, over $\mathbb{C}$, can be rephrased in terms of the radical membership problem using the *Hilbert Nullstellensatz*. Recall that, for an ideal I of an arbitrary commutative ring R, its radical ideal $\sqrt{I}$ is defined as

$$\sqrt{I} := \{h \in R \,|\, h^n \in I \text{ for some } n \in \mathbb{N}\}.$$

The following lemma gives a necessary and sufficient condition for the question in (3.5) to have a positive answer over the complex field $\mathbb{C}$.

LEMMA 3.1: *([5]) Let R be the polynomial ring $\mathbb{C}[x_1,\ldots,x_n]$. Let I be an ideal of R, and $g \in R$. Then,*

$$V(I) \subset V(g) \iff g \in \sqrt{I}$$

Unfortunately, our ground field is not $\mathbb{C}$ but $\mathbb{R}$. In this case, above lemma still provides a sufficient condition for factorability.

LEMMA 3.2: *([5]) Let R be the polynomial ring $\mathbb{R}[x_1,\ldots,x_n]$. Let I be an ideal of R, and $g \in R$. Then,*

$$g \in \sqrt{I} \;\Rightarrow V(I) \subset V(g).$$

Now the following radical ideal membership algorithm can be used for our computational test of $f \in \sqrt{I(1,1)}$.

LEMMA 3.3: *(Radical Ideal Membership, [5]) Let R be the polynomial ring $\mathbb{R}[x_1,\ldots,x_n]$. Let I be an ideal of R, and $g \in R$. Then, $g \in \sqrt{I}$ if the ideal $(I, 1-t\cdot g)$ of the ring $R[t] = \mathbb{R}[x_1,\ldots,x_n,t]$ is the unit ideal, i.e. 1 belongs to the ideal $(I, 1-t\cdot g)$.*

Remark 3.4. Note here that we have introduced a new variable t.

The ideal $(I(1,1), 1-t\cdot h) = (h_1, h_2, h_3, h_4, h_5, 1-t\cdot h)$ is a unit ideal if its Gröbner bases is $\{1\}$. Hence we can use any existing computer algebra packages to compute the Gröbner bases of the six polynomials $h_1, h_2, h_3, h_4, h_5, 1-t\cdot h$ in 9 variables.

For the Gröbner bases computation, the computer algebra package *Singular* [6] was used on a PC running Linux with 800Mz CPU and 256MB of memory.

For the above case ($d=2$), the computation gave us the positive answer in less than a second, that is, it showed that the Gröbner bases of $h_1, h_2, h_3, h_4, h_5, 1-\mathrm{t}\cdot\mathrm{h}$ is just $\{1\}$. This means that all paraunitary matrices of total degree 2 are factorable.

For higher d's, the corresponding radical ideal membership can be checked in the same fashion even though the Gröbner bases computation is much more involved.

For $d=3$, there are 4 types of Paraunitary Variety; of type $(3,0), (2,1), (1,2)$, and $(0,3)$. The $(3,0), (0,3)$ cases are trivially factorable by Corollary 2.5, and by symmetry, we have only to consider the type $(2,1)$ case. The Paraunitary Variety of type $(2,1)$ is defined by 8 quadratic polynomials in the affine 12-space, and there are 3 polynomials of degree 4 whose radical ideal membership is to be checked. The Gröbner bases computation in this case took 20 seconds, and produced $\{1\}$. Therefore, all paraunitary matrices of total degree 3 are factorable.

For the $d=4$ case, there are two nontrivial cases to consider; of type (3,1) and of type $(2,2)$. The Paraunitary Variety of type $(3,1)$ is defined by 11 quadratic polynomials in the affine 16-space, and there are 6 polynomials of degree 4 whose radical ideal membership is to be checked. For the type $(2,2)$ case, which is defined by 13 quadratic polynomials in the affine 18-space, there are 9 polynomials of degree 4 whose radical ideal membership is to be checked. The computation done in the next two sections shows that the Paraunitary Varieties of type $(3,1)$ and $(1,3)$ are completely factorable while the Paraunitary Variety of type $(2,2)$ is not. So the simplest non-factorable case should occur at type $(2,2)$.

To describe the non-factorable case more explicitly, let h_i's, $i=1,\ldots,13$ be the 13 quadratic polynomials in $\mathbb{R}[a_{ij}, b_{ij} | 0 \leq i, j \leq 2]$ that describe the Paraunitary Variety of type $(2,2)$. Let h be one of the 9 polynomials describing the factorability condition such that the ideal generated by h_i's and $1-t\cdot h$ is not the unit ideal. Then, by Hilbert Nullstellensatz, the 14 polynomials, h_i's and $1-t\cdot h$, have a common zero over $\mathbb{C}$. If one could explicitly solve the polynomial system

$$h_i = 0, \; 1 - t \cdot h = 0, \quad 1 \leq i \leq 13,$$

then the corresponding values of a_{ij}'s and b_{ij}'s will produce a non-factorable unit norm vector of type $(2,2)$.

Unfortunately, this polynomial system turned out to be too complex even for a powerful Gröbner basis package like *Singular*. A Gröbner basis package known to be extremely powerful and may suit the problems of this size is *FGb*.[1] The author was not able to access *FGb* yet, but *FGb* appears to be a promising alternative.

We will develop another method in the next section which exploits the geometrical structure of the polynomials involved. With this method, a parametrized family of non-factorable 2×2 paraunitary matrices of type $(2,2)$ is obtained. It has to be noted that, with a very powerful Gröbner basis engine, the geometric reasoning done in the next section could be automated.

4. Convex Geometric Approach

With the convex geometric approach to be taken in this section, we will actually construct the following non-factorable paraunitary matrix $\mathbf{H} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$ of type $(2,2)$ where

$$h_{11} = \frac{1}{52\sqrt{1365}}(420 + 72\sqrt{35}x + 105x^2 + 560y - 122\sqrt{35}xy - 140x^2y - 105y^2 - 60\sqrt{35}xy^2)$$

$$h_{21} = \frac{1}{52\sqrt{1365}}(30\sqrt{35}x + 105x^2 + 70y + 132\sqrt{35}xy - 770x^2y + 105y^2 + 102\sqrt{35}xy^2 + 840x^2y^2)$$

$$h_{12} = \frac{1}{52\sqrt{1365}}(-840 - 102\sqrt{35}x - 105x^2 + 770y - 132\sqrt{35}xy - 70x^2y - 105y^2 - 30\sqrt{35}xy^2)$$

$$h_{22} = \frac{1}{52\sqrt{1365}}(-60\sqrt{35}x - 105x^2 - 140y - 122\sqrt{35}xy + 560x^2y + 105y^2 + 72\sqrt{35}xy^2 + 420x^2y^2).$$

Definition 4.1.

1. For a monomial $x^i y^j$, the vector $(i,j) \in \mathbb{Z}^2$ is called its exponent vector.

2. For a Laurent polynomial $f = \Sigma\ a_{ij} x^i y^j$, the smallest convex polygon in the plane containing the exponent vectors of all the nontrivial monomial terms of f is called the convex hull of f.

To understand how the convex geometry affects the parahermitian structure, consider a polynomial vector

$$\mathbf{v} = \begin{pmatrix} f \\ g \end{pmatrix}$$
$$= \mathbf{v}_{00} + \mathbf{v}_{10}x + \mathbf{v}_{01}y, \quad \mathbf{v}_{ij} \in \mathbb{R}^2 - \{\mathbf{0}\}.$$

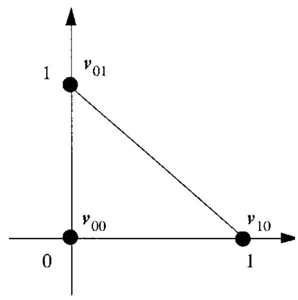

Figure 1. The convex hull of the exponent vectors of **v**.

Then the convex hull of its exponent vectors is spanned by $\{(0,0), (1,0), (0,1)\}$ (and is shown in Figure 1). Now we claim that $\mathbf{v} \in (\mathbb{R}[x,y])^2$ is not of unit norm regardless of the values of $\mathbf{v}_{ij}$'s. To see this, note that $\tilde{\mathbf{v}}\mathbf{v} = \langle \mathbf{v},\mathbf{v}\rangle = 1$ implies

$$\begin{aligned}\langle \mathbf{v}, \mathbf{v}\rangle &= \langle \mathbf{v}_{00} + \mathbf{v}_{10}\, x + \mathbf{v}_{01}\, y, \mathbf{v}_{00} + \mathbf{v}_{10}\, x + \mathbf{v}_{01}\, y\rangle \\ &= (\tilde{\mathbf{v}}_{00}\mathbf{v}_{00} + \tilde{\mathbf{v}}_{10}\mathbf{v}_{10} + \tilde{\mathbf{v}}_{01}\mathbf{v}_{01}) + (\tilde{\mathbf{v}}_{00}\mathbf{v}_{10}\, x + \tilde{\mathbf{v}}_{10}\mathbf{v}_{00}\, \frac{1}{x}) \\ &\quad + (\tilde{\mathbf{v}}_{00}\mathbf{v}_{01}\, y + \tilde{\mathbf{v}}_{01}\mathbf{v}_{00}\, \frac{1}{y}) + (\tilde{\mathbf{v}}_{01}\mathbf{v}_{10}\, \frac{x}{y} + \tilde{\mathbf{v}}_{10}\mathbf{v}_{01}\, \frac{y}{x}) = 1.\end{aligned}$$

From the fact that the last expression has to be a constant, we deduce that

$$\begin{aligned}&\mathbf{v}_{00} \perp \mathbf{v}_{10}\\ &\mathbf{v}_{00} \perp \mathbf{v}_{01}\\ &\mathbf{v}_{10} \perp \mathbf{v}_{01}.\end{aligned}$$

Quite clearly, no three nonzero vectors $\mathbf{v}_{00}, \mathbf{v}_{10}, \mathbf{v}_{01} \in \mathbb{R}^2 - \{\mathbf{0}\}$ can satisfy this mutual orthogonality.

Now we will construct a non-factorable paraunitary matrix of type $(2,2)$. And in doing this, we will actually construct a continuous one-parameter family of non-factorable paraunitary matrices of type $(2,2)$.

As we observed in (3.3), the factorability (or non-factorability) of a vector of type $(2,2)$ is characterized by the following.

$$\mathbf{v} = \sum_{i=0}^{2}\sum_{j=0}^{2} \mathbf{v}_{ij} x^i y^j \text{ is factorable} \tag{4.1}$$

$\Longleftrightarrow$ $\mathbf{v}_{00}$, $\mathbf{v}_{10}$, $\mathbf{v}_{20}$ are all parallel or $\mathbf{v}_{00}$, $\mathbf{v}_{01}$, $\mathbf{v}_{02}$ are all parallel.

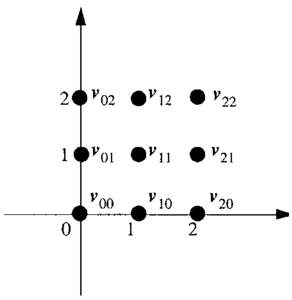

Figure 2. The exponent vectors of $\mathbf{v}$ of type (2,2).

Let

$$\mathbf{v} = \begin{pmatrix} f \\ g \end{pmatrix} = \sum_{i=0}^{2}\sum_{j=0}^{2} \mathbf{v}_{ij} x^i y^j = \sum_{i=0}^{2}\sum_{j=0}^{2} \begin{pmatrix} a_{ij} \\ b_{ij} \end{pmatrix} x^i y^j$$

be a vector of type $(2,2)$ with unit norm such that

$$\begin{aligned} &\det[\mathbf{v}_{00}, \mathbf{v}_{10}] = w,\ \det[\mathbf{v}_{10}, \mathbf{v}_{20}] = \tfrac{1}{w},\ \det[\mathbf{v}_{00}, \mathbf{v}_{20}] = 1 \\ &\det[\mathbf{v}_{00}, \mathbf{v}_{01}] = z,\ \det[\mathbf{v}_{01}, \mathbf{v}_{02}] = \tfrac{1}{z},\ \det[\mathbf{v}_{00}, \mathbf{v}_{02}] = 1, \end{aligned} \tag{4.2}$$

where w and z are nonzero real numbers. Note that, for two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$, the condition $\det[\mathbf{a}, \mathbf{b}] \neq 0$ implies that $\mathbf{a}$ and $\mathbf{b}$ are not parallel. Based on this observation and the factorability characterization in (4.1), the conditions in (4.2) ensure that the vector $\mathbf{v}$ is **non-factorable**. We will attempt to parametrize a subclass of such $\mathbf{v}$'s, thereby parametrizing a subclass of non-factorable vectors of type $(2,2)$ with unit norm.

We can find a rotation matrix $R(\theta)$ so that $R(\theta)\mathbf{v}_{00} = \begin{pmatrix} r \\ 0 \end{pmatrix}$ for some $r \in \mathbb{R}$. By replacing $\mathbf{v}$ by $R(\theta)\mathbf{v}$, if necessary, we may assume $\mathbf{v}_{00} = \begin{pmatrix} r \\ 0 \end{pmatrix}$, i.e. $a_{00} = r$ and $b_{00} = 0$.

Further assume that $\langle \mathbf{v}_{00}, \mathbf{v}_{20} \rangle = 1$ and $r \neq 0$. Then

$$\langle \mathbf{v}_{00}, \mathbf{v}_{20} \rangle = r a_{20} = 1$$

$$\det[\mathbf{v}_{00}, \mathbf{v}_{20}] = r b_{20} = 1.$$

Therefore,

$$\mathbf{v}_{20} = \begin{pmatrix} a_{20} \\ b_{20} \end{pmatrix} = \begin{pmatrix} \frac{1}{r} \\ \frac{1}{r} \end{pmatrix}.$$

Consider the A_{ij}'s defined by the expansion $\langle \mathbf{v}, \mathbf{v} \rangle = \Sigma^{2}_{i=-2}\Sigma^{2}_{j=-2} A_{ij} x^i y^j$. Then since $\langle \mathbf{v}, \mathbf{v} \rangle = 1$, one must have $A_{ij} = 0$ for any $i \neq 0$ or $j \neq 0$, and $A_{00} = 1$.

The conditions $A_{22} = \langle \mathbf{v}_{00}, \mathbf{v}_{22} \rangle = 0$ and $A_{-22} = \langle \mathbf{v}_{20}, \mathbf{v}_{02} \rangle = 0$ imply that $\mathbf{v}_{00} \perp \mathbf{v}_{22}$ and $\mathbf{v}_{20} \perp \mathbf{v}_{02}$. Hence, for some $u, v \in \mathbb{R}$

$$\mathbf{v}_{02} = u \begin{pmatrix} -b_{20} \\ a_{20} \end{pmatrix} = u \begin{pmatrix} \frac{-1}{r} \\ \frac{1}{r} \end{pmatrix},$$

$$\mathbf{v}_{22} = v \begin{pmatrix} -b_{00} \\ a_{00} \end{pmatrix} = v \begin{pmatrix} 0 \\ r \end{pmatrix}.$$

From the additional constraint det[$\mathbf{v}_{00}$, $\mathbf{v}_{02}$] = det $\begin{pmatrix} r & -\frac{u}{r} \\ 0 & \frac{u}{r} \end{pmatrix}$ =1 in (4.2), one gets $u=1$. We make a further simplifying assumption v=2, i.e. $\mathbf{v}_{02}$=$\begin{pmatrix} \frac{-1}{r} \\ \frac{1}{r} \end{pmatrix}$ and $\mathbf{v}_{22}$= $\begin{pmatrix} 0 \\ 2r \end{pmatrix}$.

By combining the assumption det[$\mathbf{v}_{00}$, $\mathbf{v}_{10}$] = w, det[$\mathbf{v}_{10}$, $\mathbf{v}_{20}$] = $\frac{1}{w}$, det[$\mathbf{v}_{00}$, $\mathbf{v}_{01}$]j = z, det[$\mathbf{v}_{01}$, v_{02}] = $\frac{1}{z}$ in (4.2) together with $A_{12}=A_{21}=A_{-12}=A_{-21}=0$, it is not hard to derive the following relations.

$$\mathbf{v}_{10} = \begin{pmatrix} a_{20} & a_{00} \\ b_{20} & b_{00} \end{pmatrix} \begin{pmatrix} w \\ \frac{1}{w} \end{pmatrix} = \begin{pmatrix} \frac{w}{r} + \frac{r}{w} \\ \frac{w}{r} \end{pmatrix}$$

$$\mathbf{v}_{01} = \begin{pmatrix} -b_{20} & a_{00} \\ a_{20} & b_{00} \end{pmatrix} \begin{pmatrix} z \\ \frac{1}{z} \end{pmatrix} = \begin{pmatrix} \frac{-z}{r} + \frac{r}{z} \\ \frac{z}{r} \end{pmatrix}$$

$$\mathbf{v}_{12} = \begin{pmatrix} b_{20} & -b_{00} \\ -a_{20} & a_{00} \end{pmatrix} \begin{pmatrix} -2w \\ \frac{1}{w} \end{pmatrix} = \begin{pmatrix} \frac{-2w}{r} \\ \frac{2w}{r} + \frac{r}{w} \end{pmatrix}$$

$$\mathbf{v}_{21} = \begin{pmatrix} -a_{20} & b_{00} \\ -b_{20} & -a_{00} \end{pmatrix} \begin{pmatrix} 2z \\ \frac{1}{z} \end{pmatrix} = \begin{pmatrix} \frac{-2z}{r} \\ \frac{-2z}{r} - \frac{r}{z} \end{pmatrix}.$$

Consider the following two constraints:

$$A_{11} = \langle \mathbf{v}_{10}, \mathbf{v}_{21} \rangle + \langle \mathbf{v}_{00}, \mathbf{v}_{11} \rangle + \langle \mathbf{v}_{11}, \mathbf{v}_{22} \rangle + \langle \mathbf{v}_{01}, \mathbf{v}_{12} \rangle = 0$$

$$A_{-11} = \langle \mathbf{v}_{10}, \mathbf{v}_{01} \rangle + \langle \mathbf{v}_{20}, \mathbf{v}_{11} \rangle + \langle \mathbf{v}_{11}, \mathbf{v}_{02} \rangle + \langle \mathbf{v}_{21}, \mathbf{v}_{12} \rangle = 0.$$

By solving these two equations $\mathrm{A}_{11} = \mathrm{A}_{-11} = 0$, one gets the following expression for $\mathbf{v}_{11}$:

$$\mathrm{v}_{11} = \begin{pmatrix} \frac{z^2(1-3r^2)-w^2(r^2-3)}{wzr} \\ \frac{wr}{2z} + \frac{3zr}{2w} \end{pmatrix}.$$

One must still consider the two constraints $A_{01}=A_{10}=0$. By an explicit calculation, one finds that

$$A_{01} = \langle \mathbf{v}_{00}, \mathbf{v}_{01} \rangle + \langle \mathbf{v}_{01}, \mathbf{v}_{02} \rangle + \langle \mathbf{v}_{10}, \mathbf{v}_{11} \rangle + \langle \mathbf{v}_{11}, \mathbf{v}_{12} \rangle + \langle \mathbf{v}_{20}, \mathbf{v}_{21} \rangle + \langle \mathbf{v}_{21}, \mathbf{v}_{22} \rangle$$

$$= \frac{(w^2 + z^2)(2r^2 - 3r^4 - 6w^2 + 5w^2 r^2)}{2w^2 z r^2}$$

$$A_{10} = \langle \mathbf{v}_{00}, \mathbf{v}_{10} \rangle + \langle \mathbf{v}_{10}, \mathbf{v}_{20} \rangle + \langle \mathbf{v}_{01}, \mathbf{v}_{11} \rangle + \langle \mathbf{v}_{11}, \mathbf{v}_{21} \rangle + \langle \mathbf{v}_{02}, \mathbf{v}_{12} \rangle + \langle \mathbf{v}_{12}, \mathbf{v}_{22} \rangle$$

$$= \frac{3(w^2+z^2)(2r^2 - r^4 - 2z^2 + 5z^2r^2)}{2wz^2r^2}.$$

Since r, w, z can take only nonzero values,

$$A_{01} = 0 \Rightarrow 2r^2 - 3r^4 - 6w^2 + 5w^2r^2 = 0 \Rightarrow w^2 = \frac{3r^4 - 2r^2}{5r^2 - 6}$$

$$A_{10} = 0 \Rightarrow 2r^2 - r^4 - 2z^2 + 5z^2r^2 = 0 \Rightarrow z^2 = \frac{r^4 - 2r^2}{5r^2 - 2}. \tag{4.3}$$

Let r be an arbitrary real number with $r > \sqrt{2}$. Let w and z be real numbers defined by

$$w = r\sqrt{\frac{3r^2 - 2}{5r^2 - 6}}$$

$$z = r\sqrt{\frac{r^2 - 2}{5r^2 - 2}}. \tag{4.4}$$

For this choice of w, z, the constraints $A_{01} = A_{10} = 0$ are automatically satisfied. And all $\mathbf{v}_{ij}$, $1 \le i, j \le 2$, can be described by a real parameter $r \in (\sqrt{2}, \infty)$. The following summarizes the resulting parametrization for a vector $\mathbf{v} = \sum_{i=0}^{2} \sum_{j=0}^{2} \mathbf{v}_{ij} x^i y^j$ of type $(2,2)$ that has unit norm and is non-factorable:

$$\mathbf{v}_{00} = \begin{pmatrix} r \\ 0 \end{pmatrix}, \ \mathbf{v}_{20} = \begin{pmatrix} \frac{1}{r} \\ \frac{1}{r} \end{pmatrix}, \ \mathbf{v}_{02} = \begin{pmatrix} \frac{-1}{r} \\ \frac{1}{r} \end{pmatrix}, \ \mathbf{v}_{22} = \begin{pmatrix} 0 \\ 2r \end{pmatrix}$$

$$\mathbf{v}_{10} = \begin{pmatrix} \frac{w}{r} + \frac{r}{w} \\ \frac{w}{r} \end{pmatrix}, \ \mathbf{v}_{01} = \begin{pmatrix} \frac{-z}{r} + \frac{r}{z} \\ \frac{z}{r} \end{pmatrix}, \ \mathbf{v}_{12} = \begin{pmatrix} \frac{-2w}{r} \\ \frac{2w}{r} + \frac{r}{w} \end{pmatrix}$$

$$\mathbf{v}_{21} = \begin{pmatrix} \frac{-2z}{r} \\ \frac{-2z}{r} - \frac{r}{z} \end{pmatrix}, \ \mathbf{v}_{11} = \begin{pmatrix} \frac{z^2(1-3r^2) - w^2(r^2-3)}{wzr} \\ \frac{wr}{2z} + \frac{3zr}{2w} \end{pmatrix}, \tag{4.5}$$

where the domain for the parameter r is $(\sqrt{2}, \infty)$, and w and z are given by the expressions in (4.4).

The normalization condition $A_{00} = 1$ has not been imposed yet. For an arbitrary fixed real number $r > \sqrt{2}$, $\beta := \mathbf{v}^t(\frac{1}{x}, \frac{1}{y})\,\mathbf{v}(x, y)$ should be a positive constant. An explicit computation shows that

$$\beta = \frac{12(5r^4 - 8r^2 + 4)}{r^2(r^2-2)(3r^2-2)(5r^2-6)(5r^2-2)}.$$

Note that, for any $r \in (\sqrt{2}, \infty)$, $\beta > 0$. Then the vector $\frac{1}{\sqrt{\beta}}\mathbf{v}$ is a desired unit norm vector that is not factorable into rotations and delays.

The parametrization given by with the additional step of normalization results in a family of non-factorable unit norm vectors $\mathbf{v} = \begin{pmatrix} f \\ g \end{pmatrix}$. This produces the family of matrices

$$\mathbf{H}(x,y) = \begin{pmatrix} f(x,y) & -x^2y^2 g\left(\frac{1}{x}, \frac{1}{y}\right) \\ g(x,y) & x^2y^2 f\left(\frac{1}{x}, \frac{1}{y}\right) \end{pmatrix}$$

w.r.t. the parameter $r \in (\sqrt{2}, \infty)$. This gives rise to a subclass of **non-factorable** paraunitary matrices of McMillan degrees $(2,2)$. The following is an explicit expression for f, g in terms of $r \in (\sqrt{2}, \infty)$:

$$\begin{aligned} f = {} & \frac{1}{\sqrt{\beta}}[r + (-\frac{z}{r} + \frac{r}{z})y - \frac{1}{r}y^2 + (\frac{w}{r} + \frac{r}{w})x \\ + {} & \frac{z^2(-3r^2+1) - w^2(r^2-3)}{wzr}xy - \frac{2w}{r}xy^2 + \frac{1}{r}x^2 - \frac{2z}{r}x^2y] \end{aligned}$$

$$\begin{aligned} g = {} & \frac{1}{\sqrt{\beta}}[\frac{z}{r}y + \frac{1}{r}y^2 + \frac{w}{r}x + (\frac{wr}{2z} + \frac{3zr}{2w})xy + (\frac{2w}{r} + \frac{r}{w})xy^2 \\ + {} & \frac{1}{r}x^2 - (\frac{2z}{r} + \frac{r}{z})x^2y + 2rx^2y^2], \end{aligned}$$

where w, z and β are given by

$$w = r\sqrt{\frac{3r^2-2}{5r^2-6}}$$

$$z = r\sqrt{\frac{r^2-2}{5r^2-2}}$$

$$\beta = \frac{12(5r^4 - 8r^2 + 4)}{r^2(r^2-2)(3r^2-2)(5r^2-6)(5r^2-2)}.$$

By specializing at $r=2$, one gets $\mathbf{H}=\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$ where

$$h_{11}=\frac{1}{52\sqrt{1365}}(420+72\sqrt{35}x+105x^2+560y-122\sqrt{35}xy-140x^2y-105y^2-60\sqrt{35}xy^2)$$

$$h_{21}=\frac{1}{52\sqrt{1365}}(30\sqrt{35}x+105x^2+70y+132\sqrt{35}xy-770x^2y+105y^2+102\sqrt{35}xy^2+840x^2y^2)$$

$$h_{12}=\frac{1}{52\sqrt{1365}}(-840-102\sqrt{35}x-105x^2+770y-132\sqrt{35}xy-70x^2y-105y^2-30\sqrt{35}xy^2)$$

$$h_{22}=\frac{1}{52\sqrt{1365}}(-60\sqrt{35}x-105x^2-140y-122\sqrt{35}xy+560x^2y+105y^2+72\sqrt{35}xy^2+420x^2y^2).$$

This is the example introduced at the beginning of this section.

5. An Application of the Convex Geometric Method

The following theorem was first observed and proved in [7]. A new proof is presented here based on the convex geometric method similar to the one used in the preceding section.

THEOREM 5.1: *The* **2×2** *paraunitary matrices of type* **(*n*,1)** or **(1,*n*)** *are completely factorable.*

Proof: Suppose that

$$\mathbf{v}=\sum_{i=0}^{1}\sum_{j=0}^{k}\mathrm{v}_{ij}x^iy^j.$$

Write $\mathbf{v}=\mathbf{v}_{i=0}(\mathrm{y})+\mathbf{v}_{i=1}(y)x$. It is easy to verify that the two face components, $\mathbf{v}_{i=0}(y)$, $\mathbf{v}_{i=1}(y)\in\mathbb{C}[y]$, of $\mathbf{v}$ are orthogonal to each other since $\mathbf{v}$ is of unit norm. This implies

$$\begin{aligned}\langle\mathbf{v}_{i=0}(y),\mathbf{v}_{i=1}(y)\rangle &=\widetilde{f_{i=0}}(y)f_{i=1}(y)+\widetilde{g_{i=0}}(y)g_{i=1}(y)\\ &=f_{i=0}(\frac{1}{y})f_{i=1}(y)+g_{i=0}(\frac{1}{y})g_{i=1}(y)\\ &=0.\end{aligned}$$

One deduces that, if $u\in\mathbb{C}$ is a zero of $f_{\mathrm{i}=0}$ but not a zero of $g_{\mathrm{i}=0}$, then $1/u$ is a zero of $g_{\mathrm{i}=1}$. There are the following two cases to consider:

1. when $f_{j=0}$ and $g_{j=0}$ have no common root
2. when $f_{j=0}$ and $g_{j=0}$ have a common root.

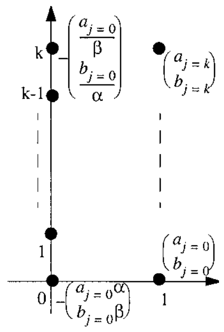

Figure 3. The convex hull of the exponent vectors of **v.**

Case 1: When $f_{j=0}$ and $g_{j=0}$ have no common root.
In this case, we can write

$$f_{j=0}(y) = a_{j=0}(y - \alpha)$$

$$g_{j=0}(y) = b_{j=0}(y - \beta), \quad \alpha \neq \beta.$$

Using the orthogonality of $\mathbf{v}_{j=0}$ and $\mathbf{v}_{j=k}$, we see that

$$f_{j=k}(y) = a_{j=k}(y - \frac{1}{\beta})$$
$$g_{j=k}(y) = b_{j=k}(y - \frac{1}{\alpha}).$$

where $a_{j=0}a_{j=k}\alpha + b_{j=0}b_{j=k}\beta = 0$. Now the convex hull generated by the exponent vectors of **v** has the structure shown in the Figure 3. Letting c_i's and d_i's, $1 \leq i \leq 1$, be the common roots of $\{f_{i=0}(y), g_{i=0}(y)\}$ and $\{f_{i=1}(y), g_{i=1}(y)\}$, respectively, one gets the following representation.

$$f_{i=0}(y) = -\frac{a_{j=k}}{\beta}(y - c_1)\cdots(y - c_l)\cdot(y - u_1)\cdots(y - u_{k-l})$$

$$g_{i=0}(y) = -\frac{b_{j=k}}{\alpha}(y - c_1)\cdots(y - c_l)\cdot(y - v_1)\cdots(y - v_{k-l})$$

$$f_{i=1}(y) = a_{j=k}(y - d_1)\cdots(y - d_l)\cdot(y - \frac{1}{v_1})\cdots(y - \frac{1}{v_{k-l}})$$

$$g_{i=1}(y) = b_{j=k}(y - d_1)\cdots(y - d_l)\cdot(y - \frac{1}{u_1})\cdots(y - \frac{1}{u_{k-l}}).$$

Now note in Figure 3 that

1. at $(0,0)$, constant coefficient of $\mathbf{v}_{j=0}$ = constant coefficient of $\mathbf{v}_{i=0}$,
2. at $(1,0)$, leading coefficient of $\mathbf{v}_{j=0}$ = constant coefficient of $\mathbf{v}_{i=1}$,

3. at $(0,k)$, constant coefficient of $\mathbf{v}_{j=\mathrm{k}}$ = leading coefficient of $\mathbf{v}_{i=0}$,
4. at $(1,k)$, leading coefficient of $\mathbf{v}_{j=\mathrm{k}}$ = leading coefficient of $\mathbf{v}_{i=1}$.

Comparing the coefficients of the polynomials involved, we get the following relations.

$$-\frac{a_{j=k}}{\beta}(-1)^k c_1\cdots c_l\cdot u_1\cdots u_{k-l} = -a_{j=0}\alpha$$

$$-\frac{b_{j=k}}{\alpha}(-1)^k c_1\cdots c_l\cdot v_1\cdots v_{k-l} = -b_{j=0}\beta$$

$$a_{j=k}(-1)^k d_1\cdots d_l\cdot\frac{1}{v_1}\cdots\frac{1}{v_{k-l}} = a_{j=0}$$

$$b_{j=k}(-1)^k d_1\cdots d_l\cdot\frac{1}{u_1}\cdots\frac{1}{u_{k-l}} = a_{j=0}.$$

Therefore, we have

$$\frac{a_{j=k}}{b_{j=k}}\frac{u_1\cdots u_{k-l}}{v_1\cdots v_{k-l}} = \frac{a_{j=0}}{b_{j=0}}.$$

Let

$$A = \frac{b_{j=k}}{\alpha}\prod_{i=1}^{l}(y-c_i)$$

$$B = (-1)^{k-l}\frac{a_{j=k}}{\prod_{i=1}^{k-l}v_i}\prod_{i=1}^{l}(y-d_i)$$

$$F = \frac{b_{j=0}}{a_{j=0}}\prod_{i=1}^{k-l}(y-u_i)$$

$$G = \prod_{i=1}^{k-l}(y-v_i).$$

Then using the above relations, one obtains

$$\mathbf{v} = \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} AF + x\cdot B\tilde{G}y^{k-l} \\ -AG + x\cdot B\tilde{F}y^{k-l} \end{pmatrix}.$$

Now $\langle \mathbf{v},\mathbf{v}\rangle = 1$ translates into

$$\begin{aligned} &A\tilde{A}F\tilde{F} + B\tilde{B}G\tilde{G} + A\tilde{A}G\tilde{G} + B\tilde{B}F\tilde{F} \\ &= (A\tilde{A} + B\tilde{B})\cdot(F\tilde{F} + G\tilde{G}) \\ &= 1. \end{aligned}$$

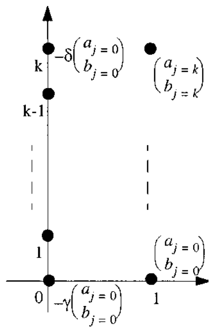

Figure 4. The convex hull of the exponent vectors of **v**.

From the last expression, we conclude that both of $A\tilde{A}+B\tilde{B}\}$ and $F\tilde{F}+G\tilde{G}$ are nonzero constants since they are involution invariant monomials. By dividing by their norms if necessary, we may assume that $A\tilde{A}+B\tilde{B}=F\tilde{F}+G\tilde{G}=1$. In this case, we have

$$\mathbf{v} = \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} AF + x \cdot B\tilde{G}y^{k-l} \\ -AG + x \cdot B\tilde{F}y^{k-l} \end{pmatrix}$$

$$= \begin{pmatrix} F & \tilde{G} \\ -G & \tilde{F} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}.$$

This shows the factorability.

Case 2: When $f_{j=0}$ and $g_{j=0}$ have a common root.
In this case, we can write

$$f_{j=0}(y) = a_{j=0}(y-\gamma)$$
$$g_{j=0}(y) = b_{j=0}(y-\gamma)$$
$$f_{j=k}(y) = a_{j=k}(y-\delta)$$
$$g_{j=k}(y) = b_{j=k}(y-\delta)$$

where $a_{j=0}a_{j=k}+b_{j=0}b_{j=k}=0$. Now the convex hull generated by the exponent vectors of **v** has the structure shown in Figure 4, and we can proceed in the same fashion as in the previous case. □

Note

1. This Gröbner basis package was developed by J.C. Faugère, and is implemented to be run over WWW. It is to be available at https://calfor.lip6.fr.

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