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Gröbner Bases for Spaces of Quadrics of Low Codimension

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Let $R = \oplus_{i\geq 0} R_i$ be a quadratic standard graded K-algebra. Backelin has shown that R is Koszul provided $\dim R_2 \leq 2$. One may wonder whether, under the same assumption, R is defined by a Gröbner basis of quadrics. In other words, one may ask whether an ideal I in a polynomial ring S generated by a space of quadrics of codimension ≤ 2 always has a Gröbner basis of quadrics. We will prove that this is indeed the case with, essentially, one exception given by the ideal $I = (x^2, xy, y^2 - xz, yz) \subset K[x, y, z]$. We show also that if R is a generic quadratic algebra with $\dim R_2 < \dim R_1$ then R is defined by a Gröbner basis of quadrics. © 2000 Academic Press

1. INTRODUCTION

A standard graded K-algebra R is an algebra of the form $R = K[x_1, \dots, x_n]/I$, where $K[x_1, \dots, x_n]$ is a polynomial ring over the field K and I is a homogeneous ideal with respect to the grading $\deg(x_i) = 1$. The algebra R is said to be *quadratic* if I is generated by quadrics (i.e., homogeneous elements of degree 2) and R is said to be *Koszul* if K admits a free resolution as an R-module whose maps are given by matrices of linear forms. For a survey on Koszul algebras and a rich bibliography we refer the reader to the recent paper of Fröberg [9]. We will just recall a few well-known facts. Koszul algebras are necessarily quadratic:

$$R \text{ is Koszul} \qquad \Rightarrow \qquad R \text{ is quadratic.}$$

We say that an ideal I in a polynomial ring is *G-quadratic* if I is generated by a Gröbner basis of quadrics with respect to some system of coordinates

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and some term order. We say that an algebra R is *G-quadratic* if its defining ideal is G-quadratic. G-quadratic algebras are Koszul:

$$R \text{ is G-quadratic} \qquad \Rightarrow \qquad R \text{ is Koszul.}$$

Neither of these implications can be reversed in general; see the examples given in [9].

Backelin proved in his thesis [1, Theorem 4.8] that if $R = \oplus_{i\geq 0} R_i$ is a quadratic standard graded K-algebra with $\dim R_2 \leq 2$ then R is Koszul. His approach is based on the study of the Koszul dual of R, which, in this case, is a non-commutative quadratic algebra with at most two relations. One may wonder whether any standard graded quadratic K-algebra R with $\dim R_2 \leq 2$ is also G-quadratic. Our goal is to prove that (under mild assumptions on the base field) this is indeed the case unless R has a very special form:

THEOREM 1. *Let $R = \oplus_{i\geq 0} R_i$ be a quadratic standard graded K-algebra with* $\dim R_2 \leq 2$. *Assume that K is algebraically closed of characteristic* $\neq 2$. *Then R is G-quadratic if and only if R is not isomorphic* (*as a graded K-algebra*) *to*

$$K[x, y, z, t_4, \ldots, t_n]/(x^2, xy, y^2 - xz, yz) + (x, y, z, t_4, \ldots, t_n)(t_4, \ldots, t_n).$$

In particular, if R is quadratic and Artinian with $\dim R_2 \leq 2$ *then it is G-quadratic.*

In the last section of the paper we show that if R is a generic quadratic algebra with $\dim R_2 < \dim R_1$ then R is G-quadratic.

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2. PRELIMINARY RESULTS

Let R be a standard graded K-algebra. If R is Artinian then it is easy to see that a necessary condition for R to be G-quadratic is that there exists a non-zero linear form x in R such that $x^2 = 0$. Extending this observation, Eisenbud et al. determined in [7, Sect. 6] a much more general obstruction to the existence of a Gröbner basis of quadrics for R in terms of the rank of the quadrics belonging to the defining ideal of R. They deduced that certain generic complete intersections of quadrics are not G-quadratic. On the other hand, a sufficient criterion for R to be G-quadratic is presented in [5] in terms of the so-called Gröbner flags. In its simplest form it says the following:

LEMMA 2. *Let $R = \oplus_{i\geq 0} R_i$ be a standard graded K-algebra and let x be a non-zero linear form in R. If $x^2 = 0$ and $xR_1 = R_2$ then R is G-quadratic (and $R_i = 0$ for $i > 2$).*

Proof. We complete x to a basis of R_1 with linear forms $x_1, x_2, \dots, x_{n-1}$ and set $x_n = x$. Then we take a presentation of $R = K[y_1, \dots, y_n]/I$, where the y_i's correspond to the x_i's. By assumption we have that I contains y_n^2 and also that it contains polynomials of the form $y_i y_j - L_{ij} y_n$ for every $1 \leq i, j < n$, where the L_{ij}'s are linear forms. Let τ be the degree reverse lexicographic order on $S = K[y_1, \dots, y_n]$ with respect to the order of the variables $y_1 > y_2 > \cdots > y_n$. By construction $\mathrm{in}(y_i y_j - L_{ij} y_n) = y_i y_j$ and hence $\mathrm{in}(I) \supseteq (y_i : 1 \leq i < n)^2 + (y_n^2)$. It follows that $\mathrm{in}(I)_2 S_1 = S_3$, where $\mathrm{in}(I)_2$ denotes the degree 2 component of $\mathrm{in}(I)$ and S_i denotes the degree i component of S. Hence $\mathrm{in}(I)$ does not contain minimal generators in degree > 2 and $R_3 = 0$. ∎

In order to apply Lemma 2 it is important to know when an algebra R has a non-zero linear form x with $x^2 = 0$.

LEMMA 3. *Let $R = \oplus_{i\geq 0} R_i$ be a standard graded K-algebra over an algebraically closed field K and let $W \subseteq R_1$ be a subspace. If* $\dim W > \dim W^2$, *then there exists a non-zero $x \in W$ such that $x^2 = 0$.*

Proof. Let $n = \dim W$ and $m = \dim W^2$. We fix bases $x_1, \dots, x_n$ and $u_1, \dots, u_m$ of W and W^2. We have relations $x_i x_j = \sum \lambda_k^{(ij)} u_k$ with $\lambda_k^{(ij)} \in K$. Now let $x = \sum a_i x_i$ be a non-zero element in W. Using the expressions above, we may write x^2 as a linear combination $\sum Q_i u_i$ of the u_i's, where the coefficients $Q_i = Q_i(a_1, \dots, a_n)$ are quadrics in the a_i. It follows that $\{x \in W : x \neq 0 \text{ and } x^2 = 0\}$ is the zero-locus of the quadrics $Q_1, \dots, Q_m$ and hence it is a projective variety of dimension $\geq (n-1) - m \geq 0$. ∎

In the proof of Theorem 1 we will need also the following reduction argument:

LEMMA 4. *Let $R = \oplus_{i\geq 0} R_i$ be a standard graded K-algebra. Let $x \in R_1$, $x \neq 0$.*

(1) *Assume that $xR_1 = 0$. Then R is G-quadratic if and only if $R/(x)$ is G-quadratic.*

(2) *Assume that x is not a zero-divisor on R. Then R is G-quadratic provided $R/(x)$ is G-quadratic.*

Proof. (1) Assume that R is G-quadratic. Then there exist a presentation of R, say $R = K[y_1, \dots, y_n]/I$, and a term order τ on $S = K[y_1, \dots, y_n]$ such that $\mathrm{in}(I)$ is generated by monomials of degree 2. We may assume that $y_1 > \cdots > y_n$ with respect to τ. Let $L = \sum_{i=1}^n \lambda_i y_i$ be the linear form of S

which corresponds to x and let k be the least integer such that $\lambda_k \neq 0$. Then we take the induced presentation $R/(x) = K[y_1, \dots, \hat{y}_k, \dots, y_n]/J$ and consider the polynomial ring $S' = K[y_1, \dots, \hat{y}_k, \dots, y_n]$ equipped with the term order τ' induced by τ. Since $Ly_i \in I$ for every i we have $y_k y_i \in \mathrm{in}(I)$ for every i. Let M be the set of the minimal monomial generators of $\mathrm{in}(I)$ which are not divisible by y_k. For every $m \in M$ we may take a quadric Q_m in I such that Q_m does not involve y_k and $\mathrm{in}(Q_m) = m$. By construction a Gröbner basis of I is given by the Q_m's and Ly_i's. It follows that the quadrics Q_m with $m \in M$ generate J. One easily sees that they are a Gröbner basis of J either by using the Buchberger algorithm or by using the fact that the Hilbert functions of R and $R/(x)$ differ only in degree 1. Similarly one proves the other implication.

(2) Let $K[y_1, \dots, y_{n-1}]/J$ be a presentation of $R/(x)$ and let $f_1, \dots, f_t$ be a Gröbner basis of J with respect to a term order τ. Let $K[y_1, \dots, y_{n-1}, z]/I$ be the induced presentation of R where z goes to x. Then the f_i's lift to polynomials $g_i = f_i + zh_i \in I$. Let σ be a term order on $K[y_1, \dots, y_{n-1}, z]$ such that $\mathrm{in}_\sigma(g_i) = \mathrm{in}_\tau(f_i)$. For instance, one can take σ to be the deg.rev.lex. product of τ with the term order of $k[z]$. By dimension arguments one proves that the g_i's are a Gröbner basis of I with respect to σ. If the f_i's are quadrics, then the g_i's are quadrics. ∎

Remark 5. With the assumption of Lemma 4, (1) or (2), it is also known that R is Koszul if and only if $R/(x)$ is Koszul; see [2, Theorem 4].

3. PROOF OF THEOREM 1

In this section we present the proof of Theorem 1. By virtue of Lemma 4 we can get rid of those linear forms $x \in R_1$ such that $xR_1 = 0$. Therefore it suffices to prove the following two propositions:

PROPOSITION 6. *Let $R = \oplus_{i \geq 0} R_i$ be a quadratic standard graded K-algebra with* $\dim R_2 \leq 2$. *Assume that K is algebraically closed of characteristic $\neq 2$ and that for every non-zero $x \in R_1$ one has $xR_1 \neq 0$. If R is not isomorphic* (*as a graded K-algebra*) *to $K[x, y, z]/(x^2, xy, y^2 - xz, yz)$ then R is G-quadratic.*

PROPOSITION 7. *Assume that the characteristic of the field K is $\neq 2$. Then the algebra $K[x, y, z]/(x^2, xy, y^2 - xz, yz)$ is not G-quadratic.*

Let us start by proving Proposition 6:

Proof. Set $n = \dim R_1$. We may clearly assume that $n > 1$. If $\dim R_2 = 0$ then the conclusion is trivial. If $\dim R_2 = 1$, then by Lemma 3 there exists a non-zero $x \in R_1$ such that $x^2 = 0$ and by assumption $xR_1 \neq 0$. Since we are assuming $\dim R_2 = 1$ it follows that $xR_1 = R_2$ and then, by Lemma 2, R is G-quadratic.

Now assume that $\dim R_2 = 2$. The case $n = 2$ is now obvious since R is a quadric hypersurface. So we may assume $n > 2$. Then by Lemma 3 there exists a non-zero $x \in R_1$ such that $x^2 = 0$. If $xR_1 = R_2$ then we are done, by Lemma 2. Otherwise if $xR_1 \neq R_2$ then $\dim xR_1 = 1$. Set $V = \{y \in R_1 : xy = 0\}$. By construction V is an $(n-1)$-dimensional subspace of R_1 and $x \in V$. Let $z \in R_1 \setminus V$ so that $R_1 = V \oplus \langle z \rangle$ and $xR_1 = \langle xz \rangle$ with $xz \neq 0$. Here we have to distinguish three cases:

Case 1: $V^2 \not\subseteq xR_1$,
Case 2: $V^2 \subseteq xR_1$ and $VR_1 \not\subseteq xR_1$,
Case 3: $VR_1 \subseteq xR_1$.

Case 1. Since $V^2 \not\subseteq xR_1$ and the characteristic of K is not 2 we may find $y \in V$ such that $y^2 \notin xR_1$. Then by construction $R_2 = \langle xz, y^2 \rangle$. Now we may complete x, y to a basis of V with elements, say, $t_3, \dots, t_{n-1}$. By a slight abuse of notation we denote again by $x, y, \dots$ the variables of the polynomial ring $S = K[x, y, t_3, \dots, t_{n-1}, z]$ that we use to present R. The equations $xV = 0$ and $R_2 = \langle xz, y^2 \rangle$ give rise to elements in the defining ideal $I \subset S$ of R,

$$\begin{array}{ll} x^2, xy, xt_3, \dots, xt_{n-1}, & \\ yt_i - a_i xz - b_i y^2 & \text{with } i = 3, \dots, n-1, \\ t_i t_j - c_{ij} xz - d_{ij} y^2 & \text{with } 3 \le i \le j \le n-1, \\ yz - exz - fy^2, & \\ t_i z - g_i xz - h_i y^2 & \text{with } i = 3, \dots, n-1, \\ z^2 - lxz - my^2, & \end{array} \qquad (*)$$

where the a_i's, b_i's, c_{ij}'s, $\dots$, m are elements in K. These polynomials define R since they are in the right number and are linearly independent. Now our strategy is the following: using the equations $(*)$ we describe the set of linear forms w in R_1 with $w^2 = 0$ and then check whether $wR_1 = R_2$. If every w fails to have $wR_1 = R_2$, then we will prove directly that the given equations in the given system of coordinates are already a Gröbner basis of I. For our goal it suffices to consider only those w of the form $w = \alpha x + \beta y + \sum_{i=3}^{n-1} \gamma_i t_i + z$, i.e., where z has coefficient 1 in w. Using the equations $(*)$ we may write w^2 as a linear combination $w^2 = pxz + qy^2$ of xz and y^2, where

$$p = 2\alpha + 2\sum_{i=3}^{n-1} \gamma_i \beta a_i + 2\beta e + \sum_{i,j=3}^{n-1} \gamma_i \gamma_j c_{ij} + 2\sum_{i=3}^{n-1} \gamma_i g_i + l$$

$$q = \beta^2 + 2\sum_{i=3}^{n-1} \gamma_i \beta b_i + 2\beta f + \sum_{i,j=3}^{n-1} \gamma_i \gamma_j d_{ij} + 2\sum_{i=3}^{n-1} \gamma_i h_i + m,$$

and where we have set $c_{ij} = c_{ji}$ and $d_{ij} = d_{ji}$ for $i > j$. It follows that $w^2 = 0$ in R if and only if $p = 0$ and $q = 0$. Note that α appears in the equation $p = 0$ with degree 1 and does not appear at all in $q = 0$. Given any set of elements $\gamma_3, \dots, \gamma_{n-1}$ we may think of $q = 0$ as an equation of degree 2 in β. Hence a solution of $q = 0$ is

$$\beta = -\left(\sum_{i=3}^{n-1} \gamma_i b_i + f\right) + \sqrt{\Delta},$$

where the discriminant is

$$\Delta = \left(\sum_{i=3}^{n-1} \gamma_i b_i + f\right)^2 - \left(\sum_{i,j=3}^{n-1} \gamma_i \gamma_j d_{ij} + 2\sum_{i=3}^{n-1} \gamma_i h_i + m\right).$$

Given β and the γ_i's the value of α is determined by the equation $p = 0$. Summing up, for every choice of $\gamma_3, \dots, \gamma_{n-1}$ we get a solution of the equations $p = 0$ and $q = 0$ of the form $w = \alpha x + \beta y + \sum_{i=3}^{n-1} \gamma_i t_i + z$. Now we check whether $wR_1 = R_2$. Note that $wx = xz$ and $wy = \mu xz + \sqrt{\Delta} y^2$ for some $\mu \in K$. It follows that $wR_1 = R_2$ provided $\Delta \neq 0$. If, for a choice of the γ_i, the corresponding Δ is non-zero, then we may conclude by Lemma 2 that R is G-quadratic. If instead for every choice of the γ_i the discriminant Δ vanishes, then Δ must be zero as a polynomial in the γ_i. This yields the following relations among the coefficients:

$$d_{ij} = b_i b_j \qquad \text{for } 3 \le i, j \le n-1,$$

$$h_i = b_i f \qquad \text{for } 3 \le i \le n-1, \qquad m = f^2.$$

These relations imply that the defining ideal I of R is contained in the ideal $(x, z - fy, t_i - b_i y : i = 3, \dots, n-1)$ and hence R has positive Krull dimension. Now let τ be the reverse lexicographic order with respect to the total order of the variables $t_{n-1} > t_{n-2} > \cdots > t_3 > z > y > x$. With respect to τ the initial terms of the polynomials $(*)$ are the terms on the left hand side. Then the initial ideal $\text{in}(I)$ of I contains the monomial ideal

$$J = (x^2, xy, yz, z^2) + (x, y, z, t_3, \dots, t_{n-1})(t_3, \dots, t_{n-1}).$$

It is easy to see that the Hilbert series $H_{S/J}(z)$ of S/J is $1 + nz + 2z^2 + z^3 + z^4 + z^5 + \cdots = 1 + nz + 2z^2 + z^3/(1-z)$. Since $\text{in}(I) \supseteq J$, $\text{in}(I)_2 = J_2$, and $S/\text{in}(I)$ has positive Krull dimension, we have $\text{in}(I) = J$; i.e., the quadrics $(*)$ are a Gröbner basis with respect to τ. This concludes the discussion of Case 1.

Case 2. Since $V^2 \subseteq xR_1$ and $VR_1 \not\subseteq xR_1$ we may find $y \in V$ such that $R_2 = \langle xz, yz \rangle$. Complete x, y to a basis of V with $t_3, \ldots, t_{n-1}$. We may now argue as in Case 1 to get the defining relations of R. In this case they are of the form

$$
\begin{array}{ll}
x^2, xy, xt_3, \ldots, xt_{n-1}, & \\
y^2 - axz, & \\
yt_i - b_i xz & \text{with } i = 3, \ldots, n-1, \\
t_i t_j - c_{ij} xz & \text{with } 3 \le i \le j \le n-1, \\
t_i z - d_i xz - e_i y^2 & \text{with } i = 3, \ldots, n-1, \\
z^2 - fxz - gy^2. &
\end{array}
\tag{$**$}
$$

Note that if we take a linear form $w = \alpha x + \beta y + \sum_{i=3}^{n-1} \gamma_i t_i + z$ then $wx = xz$ and $wy = \mu xz + yz$ with $\mu \in K$ and hence $wR_1 = R_2$. Therefore, by virtue of Lemma 2, it suffices to show that there exist $\alpha, \beta, \gamma_3, \ldots, \gamma_{n-1}$ such that $w^2 = 0$ in R. This is easy; one has just to write down the corresponding equations. This concludes the discussion of Case 2.

Case 3. Since $VR_1 \subseteq xR_1$, we have that $R_2 = \langle xz, z^2 \rangle$. The multiplication by z from V to R_2 has rank 1 and hence its kernel W has codimension 1 in V and does not contain x by construction. Let $y_2, \ldots, y_{n-1}$ be a basis of W; then $x, y_2, \ldots, y_{n-1}$ is a basis of V. Therefore the defining equations of R in this case are

$$
\begin{array}{ll}
x^2, xy_2, xy_3, \ldots, xy_{n-1}, & \\
y_2 z, y_3 z, \ldots, y_{n-1} z, & \\
y_i y_j - a_{ij} xz & \text{with } 2 \le i \le j \le n-1.
\end{array}
\tag{$***$}
$$

Note that $W^2 \subseteq \langle xz \rangle$. If $n > 3$ then by Lemma 3 there exists a non-zero $y \in W$ such that $y^2 = 0$. Set $V' = \{w \in R_1 : wy = 0\}$. By construction $x, z \in V'$ and $yR_1 = \langle xz \rangle$. Hence $(V')^2 = R_2 \not\subseteq yR_1$. This is Case 1 with y playing the role of x and hence we are done. We are left with the case $n = 3$. We have only one y_i which we denote simply by y. The defining equations are $x^2, xy, yz, y^2 - axz$. The coefficient a cannot be 0 otherwise $yR_1 = 0$, a contradiction with the assumption. Hence $a \neq 0$, and we may replace x with x/a. In the new coordinate system the defining equations are $x^2, xy, yz, y^2 - xz$. But this is the exceptional case which was excluded by assumption. This concludes the discussion of Case 3 and the proof of Proposition 6. ∎

It remains to prove Proposition 7.

Proof. We have to show that the ideal $I = (x^2, xy, yz, y^2 - xz) \subset S = K[x, y, z]$ is not generated by a Gröbner basis of quadrics, no matter what the coordinate system and the term order are. We may assume that K is algebraically closed. Unfortunately the obstruction argument of [7, Sect. 6] does not apply to this ideal. Hence we have to use another approach. Set $R = S/I$. Note that the Hilbert series of R is $1 + 3z + 2z^2 + z^3/(1 - z)$. The only quadratic monomial ideal with this Hilbert series is, up to permutation of the variables, $J = (x^2, y^2, zx, zy)$. We have:

Claim. Let D be an ideal of $K[x, y, z]$ whose initial ideal is J. Then D contains the squares of two independent linear forms.

Let us postpone for a moment the proof of the claim. Note that if I had a Gröbner basis of quadrics (in some coordinate system and with respect to some term order) then its initial ideal would be equal to J. By virtue of the claim, then, I would contain the squares of two independent linear forms. But this is a contradiction because it is easy to see that I contains (up to scalar) exactly one square of a linear form, namely the square of x.

It remains to prove the claim. To this end, let D be an ideal of $K[x, y, z]$ whose initial ideal with respect to some term order τ is J. Since x and y play exactly the same role, we may assume without loss of generality that $y < x$ with respect to τ. Then we have to discuss three cases according to how τ orders the variables:

Case 1 ($y < x < z$). The reduced Gröbner basis of D must have the form

$$y^2, x^2 - axy, xz - bxy, yz - cxy,$$

where $a, b, c \in K$. The squares of the linear forms y and $x - a/2y$ are in D.

Case 2 ($y < z < x$). The reduced Gröbner basis of D has the form

$$y^2, x^2 - axy - bz^2, xz - cxy - dz^2, yz,$$

with $a, b, c, d \in K$. Let $w = \alpha x + \beta y + \gamma z$ be a linear form. We have that w^2 is in D if and only if

$$\alpha(a\alpha + 2\beta + 2c\gamma) = 0 \qquad \text{and} \qquad \gamma^2 + 2d\alpha\gamma + b\alpha^2 = 0.$$

It follows that there are at least two squares of independent linear forms in D, namely the squares of y and $x + \beta y + \gamma z$, where $\gamma = -d + \sqrt{d^2 - b}$ and $\beta = -a/2 + c\gamma$.

Case 3 ($z < y < x$). The reduced Gröbner basis of D has the form

$$y^2 - az^2, x^2 - bxy - cz^2, xz - dz^2, yz - ez^2,$$

with $a, b, c, d, e \in K$. Let $w = \alpha x + \beta y + \gamma z$ be a linear form. We have that w^2 is in D if and only if

$$b\alpha^2 + 2\alpha\beta = 0 \qquad \text{and} \qquad c\alpha^2 + a\beta^2 + \gamma^2 + 2d\alpha\gamma + 2e\beta\gamma = 0.$$

Then there are at least two squares of independent linear forms in D, one with $\alpha = 0$ and $\beta = 1$ and another one with $\alpha = 1$.

This concludes the proof of Proposition 7.

A description of the possible Hilbert series of quadratic algebras with $\dim R_2 \leq 2$ has been given by Backelin in [1, Theorem 4.8]. One can get this description also as a by-product of (the proof of) Theorem 1. One can obtain also a short of "normal form" for the algebras of this class. For instance:

EXAMPLE 8. Let K be an algebraically closed field of characteristic $\neq 2$. Let R be a quadratic Artinian standard graded K-algebra with $\dim R_2 = 2$. Then the Hilbert series of R is $H_R(z) = 1 + nz + 2z^2$. Moreover the algebra R is defined by an ideal of the form $(x_1^2, x_1x_2, \ldots, x_1x_{n-2}) + (x_ix_j - a_{ij}x_1x_{n-1} - b_{ij}x_1x_n : 2 \leq i \leq j \leq n)$, where $a_{ij}, b_{ij} \in K$ and the given equations are a Gröbner basis w.r.t. the reverse lexicographic order induced by $x_n > x_{n-1} > \cdots > x_1$.

The algebra $R = K[x, y, z]/(x^2, xy, y^2 - xz, yz)$ is essentially the only quadratic algebra with $\dim R_2 = 2$ which is not G-quadratic. By virtue of Backelin's theorem we know that it is Koszul. Indeed one can even show that there exists a Koszul filtration for R (see [4] for the definition of Koszul filtration). The filtration is given by $\mathbf{F} = \{(0), (y), (z), (z, x), (z, x, y)\}$. To check that $\mathbf{F}$ is a Koszul filtration one has to check that

$$0 : y = (x, z), \qquad 0 : z = (y),$$
$$(z) : x = (x, y, z), \qquad (z, x) : y = (x, y, z)$$

and this is easy.

An interesting corollary of Theorem 1 is:

COROLLARY 9. *Let R be a quadratic Cohen–Macaulay standard graded K-algebra with h-vector $(1, h_1, h_2, h_3, \ldots, h_s)$. Assume that K is algebraically closed of characteristic $\neq 2$ and that $h_2 \leq 2$. Then R is G-quadratic and $h_i = 0$ for $i > 2$.*

Proof. This follows immediately from Lemma 4 and Theorem 1. ∎

4. THE GENERIC CASE WITH $\dim R_2 < \dim R_1$

Let R be a quadratic K-algebra and assume that $\dim R_2 < \dim R_1$. Then by Lemma 3 there exists a non-zero $x \in R_1$ such that $x^2 = 0$. If $xR_1 = R_2$ then by Lemma 2 we may conclude that R is G-quadratic. Since we are assuming $\dim R_2 < \dim R_1$, there are no dimensional obstructions to the equality $xR_1 = R_2$. Therefore if R is "generic" enough then one expects that R is G-quadratic. Let us try to make this precise.

Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring and let m be an integer. For every space of quadrics $V \subseteq S_2$ of codimension m we get a quadratic algebra $R = S/(V)$; here (V) denotes the ideal generated by the elements in V. The set of the spaces of quadrics of codimension m is a projective variety, the Grassmannian $\mathrm{Grass}(m, S_2)$, which is embedded in the projective space $\mathbf{P}^N$ via the Plücker map. Here

$$N = \binom{\dim S_2}{m} - 1.$$

Therefore the family of the quadratic algebras with $\dim R_1 = n$ and $\dim R_2 = m$ gets identified with the Grassmannian $\mathrm{Grass}(m, S_2)$ via the correspondence $V \to S/(V)$. We then say that a property P holds for a generic algebra if there exists a non-empty Zariski open U of $\mathrm{Grass}(m, S_2)$ such that for every $V \in U$ the algebra $R = S/(V)$ has property P.

THEOREM 10. *Let K be an algebraically closed field. Then a generic quadratic K-algebra R with* $\dim R_1 > \dim R_2$ *is G-quadratic.*

Proof. Set $n = \dim R_1$ and $m = \dim R_2$. By virtue of Lemma 2 it suffices to prove that the property "there exists a non-zero $x \in R_1$ such that $x^2 = 0$ and $xR_1 = R_2$" holds in a non-empty Zariski open subset of $\mathrm{Grass}(m, S_2)$. To this end, let us consider the following sets associated with $V \in \mathrm{Grass}(m, S_2)$:

$$X_V = \{x \in S_1 \mid x \neq 0, x^2 \in V\}, \qquad Y_V = \{x \in S_1 \mid x \neq 0, xS_1 + V \neq S_2\}.$$

We want to show that there exists a non-empty open subset U of $\mathrm{Grass}(m, S_2)$ such that for every $V \in U$ one has $X_V \not\subseteq Y_V$. Fix a basis $x_1, \ldots, x_n$ of S_1 and consider the induced monomial basis $x_i x_j$ in S_2. Any $V \in \mathrm{Grass}(m, S_2)$ is determined by a matrix A whose rows correspond to a basis of V and whose maximal minors give the Plücker coordinates of V in $\mathbf{P}^N$. Let $x = \sum \alpha_i x_i \in S_1$. Note that the condition $x^2 \in V$ can be translated into the vanishing of the maximal minors of the matrix obtained from A by adding the row of the coefficients of x^2. If we take V in the open subset of the Grassmannian given by the non-vanishing of one Plücker coordinate, then it suffices to consider only m of the maximal minors of the extended matrix. Therefore X_V is the zero-set of polynomials

$f_1, \dots, f_m$ in the α_i's of degree 2 whose coefficients are the Plücker coordinates of V. Similarly the condition $xS_1 + V \neq S_2$ can be translated into a set of polynomial equations $g_1, \dots, g_b$ in the α_i whose number and degrees do not depend on V and whose coefficients are polynomials in the Plücker coordinates of V. If $X_V \subseteq Y_V$ then by Hilbert Nullstellensatz one has $(g_1, \dots, g_b) \subseteq \sqrt{(f_1, \dots, f_m)}$. By the effective version of the Nullstellensatz (see [11]) there exists an integer k, not depending on V, such that $g_i^k \in (f_1, \dots, f_m)$ for every i. In our case the best known bound for k is $k = 2^{n+1}$; it follows from Sombra's result [14, Theorem 3.19] by applying the so-called Dube trick. We claim that for a generic V the polynomials $f_1, \dots, f_m$ form a regular sequence. By the semicontinuity of the fiber dimension (see for instance [6, Corollary 14.19]) it is enough to exhibit a space $V \in \mathrm{Grass}(m, S_2)$ such that the codimension of X_V is m. Let us postpone this for a moment. We may assume that $f_1, \dots, f_m$ is a regular sequence and hence the Hilbert function of the ideal $(f_1, \dots, f_m)$ is the largest possible. The conditions $g_i^k \in (f_1, \dots, f_m)$ can be translated into polynomial relations among the coefficients of the f_i and g_i and hence into polynomial relations among the Plücker coordinates of V. Now it suffices to show that these polynomial relations are non-trivial. To this end it is enough to give an example of V such that $X_V \not\subseteq Y_V$ and $f_1, \dots, f_m$ are a regular sequence. Here is the example: let V be the space of quadrics of $K[x_1, \dots, x_n]$ generated by

$$\begin{array}{ll} x_j x_n & \text{with } m < j \leq n, \\ x_i x_j & \text{with } 1 \leq i < j \leq n-1, \\ x_j^2 - x_j x_n & \text{with } 1 \leq j \leq n-1. \end{array}$$

By construction one has $x_n \in X_V \setminus Y_V$. The polynomials $f_1, \dots, f_m$ are in this case $f_1 = \alpha_1^2 + 2\alpha_1\alpha_n$, $f_2 = \alpha_2^2 + 2\alpha_2\alpha_n, \dots, f_m = \alpha_m^2 + 2\alpha_m\alpha_n$ and they clearly form a regular sequence. ∎

One may wonder whether one can replace "generic" with Artinian in Theorem 10. As we have seen, this is the case if $\dim R_2 = 2$. But in general it is not possible; there exist Artinian quadratic algebras with $\dim R_1 > \dim R_2$ which are not G-quadratic. Here is an example:

EXAMPLE 11. Let $R = K[x_1, \dots, x_4]/I$, where I is generated by six generic quadrics. The Hilbert series of R is $1 + 4z + 4z^2$ and R is not G-quadratic because its defining ideal does not contain the square of a linear form. Set $A = R[t]/(t)(x_1, \dots, x_4, t)$. Then A has Hilbert series $1 + 5z + 4z^2$ and by Lemma 4 we know that A is not G-quadratic. Explicitly, over a field of characteristic $\neq 2$, say, one can take $I = (x_1^2 + x_2x_3,\ x_2^2 - x_3x_4,\ x_3^2 - x_4x_1,\ x_4^2 - x_1x_2,\ x_1x_3,\ x_2x_4)$.

This example is not completely satisfactory because the ring A has a socle element in degree 1, namely t. One would like to see an example of a quadratic but not G-quadratic algebra R such that $\dim R_1 > \dim R_2$, R is Artinian (or even with $R_3 = 0$), and $xR_1 \neq 0$ for every non-zero $x \in R_1$. Here is a candidate:

EXAMPLE/PROBLEM 12. Let $R = K[x, y_1, y_2, z]/I$, where I is generated by the following seven quadrics:

$$x^2, xz, y_1^2, y_1z, y_2^2, y_2z, z^2 - xy_1 - xy_2.$$

The Hilbert series of R is $1 + 4z + 3z^2$ and it is easy to check that there are no socle elements in degree 1. It is not difficult to see that the ideal I does not have a Gröbner basis of quadrics in the given coordinate system. We have also checked (with CoCoA [3]) that there is no element $x \in R_1$ with $x^2 = 0$ and $xR_1 = R_2$ and also that R does not have a Gröbner flag (see [5] for the definition). This is our candidate to be the algebra with the above mentioned properties. Unfortunately we are not able to show that R is not G-quadratic but we believe that it is so. Note that, according to Roos' table [13], the algebra R is Koszul. One can show this also by checking that

$$\mathbf{F} = \{0, (y_1), (y_1, z), (y_1, y_2), (y_1, x), (y_1, y_2, z), \\ (y_1, y_2, x), (y_1, z, x), (y_1, y_2, z, x)\}$$

is a Koszul filtration for R.

A few remarks are in order about the problem above and its generalization. Let I be a homogeneous ideal in a polynomial ring $S = K[x_1, \ldots, x_n]$. Any invertible matrix $g = (g_{ij}) \in \mathrm{GL}_n(K)$ acts on S as a graded K-isomorphism by setting $g(x_j) = \sum_i x_i g_{ij}$ (and any graded K-isomorphism of S is of this type). Consider the set $\mathrm{In}(I)$ of all the initial ideals $\mathrm{in}_\tau g(I)$ of the ideal I, as g varies in $\mathrm{GL}_n(K)$ and τ varies in the set of the term orders. What can be said about $\mathrm{In}(I)$? Can one determine it algorithmically? If we fix g and let τ vary, then the Gröbner fan of I answers the questions; see [12; 15, Chap. 2]. If instead we fix τ and let g vary, then it is known that for "almost all" the g one obtains the same initial ideal, the generic initial ideal of I w.r.t. τ; see for instance [6, Sect. 15.9]. In general one has

LEMMA 13. *The set* $\mathrm{In}(I)$ *is finite.*

Proof. The set $\mathrm{In}(I)$ is contained in the set $H(I)$ of the monomial ideals whose Hilbert function is equal to that of I. The set $H(I)$ is finite. This follows (for instance) from a result of Elias et al. [8] which says that the degrees of the minimal generators of an ideal I are bounded by a number, say s, which depends only on the Hilbert function of I. ∎

The number s can be determined explicitly; it is the highest degree of a minimal generator of the lex segment ideal with the given Hilbert function. Therefore, since there are finitely many degrees involved in the process and hence only finitely many distinct term orders, the problem of determining $\mathrm{In}(I)$ boils down to:

Problem 14. Given a space V of forms of degree d and a term order τ determine the set of monomial spaces $\{\mathrm{in}_\tau g(V) \mid g \in GL_n(K)\}$.

It is in principle clear how to proceed: just reduce the space $g(V)$ to the "echelon form" with respect to τ. The result of the process depends on the vanishing or non-vanishing of some polynomial equations in the entries of g. Hence it gives rise to a stratification of $\mathrm{GL}_n(K)$ describing the various $\mathrm{in}_\tau g(V)$. Whether this can be done in an efficient way is another story. A useful observation is that if $x_1 > x_2 > \cdots > x_n$ w.r.t. to τ and b is a lower triangular invertible matrix then for every $f \in S$ one $\mathrm{in}_\tau b(f) = \mathrm{in}_\tau(f)$ up to scalar. Hence $\mathrm{in}_\tau bg(V) = \mathrm{in}_\tau g(V)$ for every g. Therefore only the flag associated to g counts, not g itself.

Recently Fröberg and Löfwall [10] announced that they have shown that a generic space of quadrics V of $S = K[x_1, \ldots, x_n]$ defines a Koszul algebra if and only if $\dim V \le n$ (the complete intersection case) or $\dim V \ge \dim S_2 - n^2/4$.

One may ask when a generic space of quadrics V of $S = K[x_1, \ldots, x_n]$ defines a G-quadratic ideal (V). Assume that K is algebraically closed of characteristic $\neq 2$. Set $d = \dim V$. The case $d = 1$ is trivial. If $d = 2$, then it follows from Corollary 9 that (V) is G-quadratic. Assume from now on that $d \ge 3$. Eisenbud et al. [7, Corollary 20] shown that if $d \le n$ (the complete intersection case) and $n < (d^2 + 3d + 2)/6$ then (V) is not G-quadratic. On the other hand, if $d \ge n$ (the Artinian case) one has that (V) is G-quadratic if and only if $d > \dim S_2 - n$: this follows from Theorem 10 and from the fact that if $\dim S_2 - n \ge d \ge n$ then V does not contain a square of a linear form. So we are left with the cases $d \ge 3$ and $(d^2 + 3d + 2)/6 \le n$. We do not know what the behavior is in this range. The first open case is $d = 3$ and $n = 4$, i.e., a generic complete intersection of three quadrics in four variables.

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