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GRÖBNER–SHIRSHOV BASES OF THE LIE ALGEBRA B_n^+

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Abstract. The minimal Gröbner–Shirshov bases of the positive part B_n^+ of a simple finite-dimensional Lie algebra B_n over an arbitrary field of characteristic 0 are calculated, for the generators associated with simple roots and *for an arbitrary ordering* of these generators (i.e., an arbitrary basis of the $n!$ Gröbner–Shirshov bases is chosen and studied). This is a completely new class of problems; until now, this program was carried out only for the Lie algebra A_n^+. The minimal Gröbner–Shirshov basis of the Lie algebra B_n^+ was calculated earlier by Bokut and Klein, but this was done for only one ordering of generators.

§1. Minimal Gröbner–Shirshov bases of Lie algebras

We fix an arbitrary finite linearly ordered set X of letters. In what follows we tacitly assume that all words are composed of elements of X. We introduce a linear order *lex* on the set of all associative words in X; this order differs from the lexicographic ordering only in one respect: any proper beginning of a word is greater than the word itself.

As in [1, 2], we say that an associative word is *regular* if it is greater than any of its proper endings.

In [1], it was shown that for an arbitrary regular word u there exists a unique nonassociative word (u) that differs from u only in the presence of brackets and satisfies the following conditions: if u is not a letter, then $(u) = (u_1)\cdot(x)$, where u_1 and x are regular words; if in this situation u_1 is not a letter, then $(u_1) = (y)\cdot(z)$, where y, z are regular words and $z \leq x$. The mapping that associates the word (u) with any regular word u will be called the *regular arrangement of brackets*.

All spaces and algebras that appear below will be spaces and algebras over an arbitrary field k of characteristic 0. Let $\mathrm{Lie}(X)$ be the free Lie algebra generated by the set X. For a regular word u, denote by $[u]$ the element of the algebra $\mathrm{Lie}(X)$ obtained by application of the regular arrangement of brackets and the subsequent canonical mapping to the algebra $\mathrm{Lie}(X)$. We introduce the following notation: $k\,\mathrm{Reg}(X)$ is the space with the basis consisting of all regular words; $[\cdot] : k\,\mathrm{Reg}(X) \to \mathrm{Lie}(X)$ is the linear mapping that sends any regular word u to the element $[u]$.

In [1], Shirshov showed that the set of all elements of the form $[u]$, where u runs through the set of all regular words, is a basis of the algebra $\mathrm{Lie}(X)$. It follows that the mapping $[\cdot]$ is bijective. By the *support* of an element x of $\mathrm{Lie}(X)$ we mean the inverse image of x under the mapping $[\cdot]$. In what follows, it will always be assumed that all words are associative.

Consider an arbitrary finitely generated Lie algebra $L = \mathrm{Lie}(X)/J$ given by its generators (elements of X) and relations (elements of the Lie ideal J of $\mathrm{Lie}(X)$). Usually,

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a *relation* of L is understood as a formula $u = 0$, where u is an arbitrary element of J. But for us it will be convenient to refer to the elements $u \in J$ themselves as relations of L. A relation u is said to be *nontrivial* if u is nonzero.

We denote by I the inverse image of J under the mapping $[\cdot]$ and introduce the *deg-lex* order on the set of associative words: first the lengths of words are compared, and the words of the same length are ordered lexicographically.

A regular word u will be called a *reduced word* if there is no element $h \in k\,\mathrm{Reg}(X)$ such that $u - h \in I$ and h is a linear combination of regular words that are smaller than u (with respect to the *deg-lex* order). Obviously, the elements of the form $[u] + J$, where u runs through the set of all reduced words, constitute a basis of the algebra $\mathrm{Lie}(X)/J$. As in [3, 4], we shall say that such a basis is *reduced*.

Remark 1.1. It is clear that each relation is a linear combination of relations of the form $[u-\pi(u)]$, where u is a regular nonreduced word and $\pi(u)$ is a uniquely determined linear combination of reduced words. We shall say that such relations are *canonical*.

Let h be a nonzero element of the space $k\,\mathrm{Reg}(X)$; a regular word u is called the *highest word* of h if there exists a nonzero element α of the field k such that the element $h - \alpha u$ is a linear combination of regular words that are smaller than u (with respect to the *deg-lex* order). Clearly, the highest word of h is unique; we shall denote it by $\overline{h}$. For an arbitrary set S of nonzero elements of $k\,\mathrm{Reg}(X)$, we denote by $\overline{S}$ the set of the highest words of the elements of S.

A set S of nontrivial relations of the Lie algebra $L = \mathrm{Lie}(X)/J$ is called a *Gröbner–Shirshov basis*, or GShB, of the Lie algebra L if S generates the ideal J and is closed under compositions; see [5]. The following lemma by Shirshov is well known: a set S of nontrivial relations generating the Lie ideal J is a GShB if and only if the highest word of the support of each nontrivial relation has a subword that is the highest word of the support of a relation belonging to S.

In [3, §2] it was shown that the intersection of all GShBs of the canonical relations of the algebra $L = \mathrm{Lie}(X)/J$ is a GShB of this algebra. This intersection is called the *minimal GShB*, or MGShB [3, §2].

Remark 1.2. The MGShB of a finitely generated Lie algebra $\mathrm{Lie}(X)/J$ coincides with the set of all canonical relations for which the highest word of their support is not reduced, but each its proper regular subword is reduced; see [3, §2].

This is exactly our working definition of an MGShB, and there is no need to recall the definition of compositions in this paper. Remark 1.2 shows that for the calculation of the MGShB it remains to learn to calculate nonreduced regular words such that every proper regular subword of them is reduced, and also to construct canonical relations with such highest words.

Now, we turn to the case where the ideal J is homogeneous with respect to the number of occurrences of each letter. Let Z denote the free Abelian group generated by the set X. The operation in this group will be denoted additively. The algebra $\mathrm{Lie}(X)$ is graded by the group ZX. The graduation is defined by the following condition: a generator $x \in X$ is homogeneous as an element of the algebra $\mathrm{Lie}(X)$, and it belongs to the homogeneous component corresponding to x viewed as an element of ZX. This graduation induces a graduation on the algebra $L = \mathrm{Lie}(X)/J$.

The image of a word u under the natural homomorphism of the semigroup of words to the group ZX will be denoted by $|u|$, and will be called the *composition of* u. If the homogeneous component of the algebra L that corresponds to $|u|$ is not zero, we shall say that the composition of the word u is *rooted*, or that the word u has *rooted composition.* Now we recall some useful properties of reduced words.

Lemma 1.1. *Let $L = \mathrm{Lie}(X)/J$ be a finitely generated algebra such that the ideal of its relations is homogeneous with respect to the number of occurrences of each letter. Then*:

1) *the composition of any reduced word is rooted*;
2) *each regular subword of a reduced word is reduced*;
3) *the compositions of all regular subwords of a reduced word* (*including the word itself*) *are rooted.*

Proof. Statement 1) follows immediately from the definitions, statement 2) follows from [1, Lemma 4], and statement 3) is a consequence of statements 1) and 2). □

As in [3, 4], by an RR-*word* we mean any regular word such that the compositions of all its regular subwords (including the word itself) are rooted.

Remark 1.3. Let the assumptions of Lemma 1.1 be fulfilled. Consider the set of all RR-words the compositions of which are equal to a given element of the group ZX. Then the smallest of these RR-words is reduced.

Now we consider a particular case: assume that every RR-word of the algebra $L = \mathrm{Lie}(X)/J$ is uniquely determined by its composition (later we shall show that this assumption is fulfilled for the algebra B_n^+).

A regular word will be called an SR-*word* if its composition is not rooted, but the compositions of all its proper regular subwords are rooted.

Lemma 1.2. *Let $L = \mathrm{Lie}(X)/J$ be a finitely generated Lie algebra such that* (i) *its ideal of relations is homogeneous with respect to the number of occurrences of each letter*; *and* (ii) *every* RR-*word is uniquely determined by its composition. Then*:

1) *the set of reduced words coincides with the set of* RR-*words*;
2) *the set of regular nonreduced words with all proper regular subwords reduced coincides with the set of* SR-*words*;
3) *the MGShB of L coincides with the subset of the free Lie algebra* $\mathrm{Lie}(X)$ *formed by the elements* $[u]$*, where u runs through the set of all* SR-*words.*

Proof. Statement 1) follows from statement 3) of Lemma 1.1 and Remark 1.3, and statement 2) follows from Remark 1.2 and statement 1).

3) Using Remark 1.2 and statement 1), we see that the MGShB of L coincides with the set of relations of the form $[u - \pi(u)]$, where u is an SR-word and $\pi(u)$ is a uniquely determined linear combination of reduced words. Since u is an SR-word, its composition is not rooted, which means that the homogeneous component of the algebra L that corresponds to $|u|$ consists only of 0. Therefore, we have $\pi(u) = 0$ and $[u - \pi(u)] = [u]$, as desired. □

In this paper, our aim is to calculate the MGShB of the Lie algebra B_n^+ for generators corresponding to simple roots. For this, we shall calculate the set of RR-words of the algebra B_n^+ for any ordering of generators; next, we shall show that any RR-word is uniquely determined by its composition and shall calculate the set of SR-words. The main result of this paper is Theorem 9.2.

In [3], this program was carried out for the algebra A_n^+ and for any ordering of its set of generators. For the algebra B_n^+, the reduced words were calculated in the papers [6, 7], and the MGShB was calculated in [7], but for only one ordering of generators.

§2. The algebra B_n^+, the root system, and graphs of positive roots

To describe the algebra B_n^+ and its generators corresponding to simple roots, we recall some well-known facts from the classical theory of finite-dimensional simple Lie algebras over a field of characteristic 0; see [8, 9].

The algebra B_n is a simple finite-dimensional decomposable Lie algebra. Let L be any finite-dimensional simple decomposable Lie algebra. The term *decomposable* means that L contains a nilpotent subalgebra H (Cartan subalgebra) such that the Lie H-module L is the direct sum of H and the submodules

$$L_\alpha = \{x \in L \mid \exists n \geq 1\ \forall h \in H\ (\mathrm{ad}_h - \alpha(h)\cdot 1)^n \cdot x = 0\},$$

where α runs through the set of nonzero weights of H-semiinvariants of L.

The nonzero weights of H-semiinvariants are called *roots.* By this definition, the roots are elements of the space H^* dual to H. The Cartan subalgebra is commutative. The spaces L_α are 1-dimensional, and their elements are H-semiinvariants.

There exists a subset Π of the set of roots such that any root α can be uniquely represented in the form $\alpha = \sum a_\pi \cdot \pi$, where the sum is taken over some (not necessarily all) elements π of Π and the coefficients a_π are nonzero integers having one and the same sign. Such a subset Π is called a *system of simple roots.* The roots that are sums of simple roots (as elements of the space H^*) are said to be *positive.*

Simple roots are linearly independent elements of the H^*; it follows that the subgroup $Z\Pi$ of the additive group of H^* generated by simple roots is the free Abelian group generated by simple roots. Each root is an element of $Z\Pi$. The algebra L is graded by the group $Z\Pi$. In this graduation any homogeneous component corresponding to a root is 1-dimensional.

We denote by L^+ the subspace of L generated by all components L_α corresponding to positive roots α. Then L^+ is a subalgebra of L, and it is natural to call it the "positive part" of L. This subalgebra is generated by the H-semiinvariants the weights of which are simple roots. For each simple root π, choose a nonzero H-semiinvariant of the weight π. The set X of these elements generates the algebra L^+. The elements of X are precisely the generators that are of interest for us.

The algebra B_n^+ is "the positive part" of the algebra B_n; observe that n is the number of simple roots of the algebra B_n.

Remark 2.1. The relations of the algebra L are homogeneous with respect to the number of occurrences of each letter. Therefore, the algebra L is graded by the group ZX. Observe that the set of simple roots is in one-to-one correspondence with the set X of generators of the algebra L^+. It follows that the graduation of L by the group $Z\Pi$ coincides with the graduation by ZX. Hence, to know what elements of $Z\Pi$ are positive roots is the same as to know what homogeneous components of the algebra L^+ are not equal to 0.

Remark 2.2. Assume that we know the ordering of generators and that we know also what elements of the group $Z\Pi$ are positive roots (i.e., what homogeneous components of the algebra L^+ are not equal to 0). It is interesting to note that this is the only information that we need to calculate the MGShB of the algebra B_n^+.

Indeed, this information suffices for the calculation of the sets of RR-words and SR-words of the algebra L^+. As we shall show later, the RR-words are uniquely determined by their compositions. Hence, by Lemma 1.2, we can calculate the MGShB of L^+.

Remarks 2.1 and 2.2 show that it is not necessary to distinguish between letters and the simple roots corresponding to these letters. But if it is important to stress that we view a letter q as a simple root, we write $|q|$ instead of q.

Let ZX be the free Abelian group for which X is the set of free generators; we say that an element of ZX is *positive* if it is a sum of elements of X. We define a partial order on the group ZX as follows: $\alpha < \beta$ if $\beta - \alpha$ is positive.

Now we are ready to introduce the following general notion, which is a starting point of our investigation.

By a *set of data* we mean a triple $\langle X, \Phi^+, \leq\rangle$, where $\langle X, \leq\rangle$ is a finite linearly ordered set and Φ^+ is a collection of positive elements of the group ZX.

The elements of X are called *letters*, or *simple roots*, and the elements of Φ^+ are *positive roots*.

The most transparent description of the set of data for the algebra B_n^+ can be given in terms of graphs.

By a *graph* we mean a pair $\langle X, R\rangle$ that consists of a finite set X (the set of vertices) and a symmetric antireflexive binary relation R on X.

For a graph D, we denote by $\sum D$ the element of ZX equal to the sum of all vertices of the graph. Below, we usually denote the sum of the vertices of a graph by the same symbol as the graph itself (if this leads to no confusion).

A *graph with ordering of vertices* is a triple $\langle X, R, \leq\rangle$ such that $\langle X, R\rangle$ is a graph and $\leq$ is a linear ordering of the set X.

We say that two vertices x, y of a graph are *adjacent* if they are connected by an edge, i.e., $(x, y) \in R$. A graph is *connected* if for any two vertices x, y there exists a finite sequence of vertices $x_1, \dots, x_m$ such that $x = x_1$, $y = x_m$, and the vertices x_i, x_{i-1} are adjacent for all $i = 1, \dots, m-1$.

A (finite) connected graph is said to be an *interval* if each vertex has at most two adjacent vertices, and at least one vertex has at most one adjacent vertex. A vertex of an interval with at most one adjacent vertex is called an *end vertex*.

Any (finite) interval I either consists of only one vertex or has exactly two end vertices. Any interval I that is a subgraph of a graph D without cycles is uniquely determined by its end vertices x, y; in this case we write $I = [x, y]$. If t is a vertex that is not contained in the interval I but is adjacent to y, we denote I by $[x, t)$ or $(t, x]$. Finally, we denote the interval I by (s, t) if s, t are vertices of D that are not contained in I, but s is adjacent to x and t is adjacent to y.

We return to the set of data for the algebra B_n^+. Consider the graph the vertices of which are simple roots of B_n, and the vertices α, β are adjacent if the element $\alpha + \beta$ of the group $Z\Pi$ is again a root. Using the tables given at the end of the book [10], we can conclude that this graph is an interval $[z_1, z]$, where z_1 is the only simple root such that z_1 has only one adjacent simple root and $2z_1 \nleq \alpha$ for all roots α, i.e., $\alpha \neq 2z_1$ and $\alpha - 2z_1$ is not a sum of simple roots. Moreover, z is the only simple root that has only one adjacent simple root and is not equal to z_1.

Remark 2.3. The table given at the end of the book [10] shows that the set of positive roots of B_n, viewed as elements of the group $Z\Pi$, consists of all roots that have one of the following forms:

$\sum[\alpha, \beta]$ (the sum of all vertices belonging to the interval $[\alpha, \beta]$), or

$\sum[\alpha, z] + \sum[\beta, z]$ $(\alpha \neq \beta)$.

Here $[\alpha, \beta]$ is a subinterval of the interval $[z_1, z]$ introduced above.

Definition 2.1. By a *set of data for* B_n $(n \geq 2)$ we mean a quintuple $\langle \Pi, R, z, B_n^+, \leq\rangle$ in which Π is a set consisting of n elements; $\leq$ is a linear ordering of Π; $\langle \Pi, R\rangle$ is a graph, and moreover, an interval, for which z is one of the end vertices; and B_n^+ is the set of all elements of the group $Z\Pi$ that have one of the following forms: $\sum[\alpha, \beta]$ $(\alpha, \beta \in \Pi)$ or $\sum[\alpha, z] + \sum[\beta, z]$ $(\alpha, \beta \in \Pi,\ \alpha \neq \beta)$.

By the *graph of* B_n we mean the quadruple $\langle \Pi, R, z, \leq\rangle$. This is a graph with ordering of vertices and with a distinguished vertex z. For a given set of data for B_n, the triple $\langle \Pi, B_n^+, \leq\rangle$ is a set of data, and the graph of B_n determines this set of data completely.

Definition 2.2. Let α be a positive root from the set of data for B_n. By the *graph* $D(\alpha)$ we mean the quadruple $\langle\overline{\Pi}, \overline{R}, f, d\rangle$ in which

- $\overline{\Pi}$ is the set of simple roots β such that $\beta \le \alpha$;
- $\langle\overline{\Pi}, \overline{R}\rangle$ is the graph obtained from the graph $\langle\Pi, R\rangle$ by elimination of vertices that do not belong to $\overline{\Pi}$;
- f is the canonical embedding of $\overline{\Pi}$ in the set Π;
- d is the mapping that sends any element β of $\overline{\Pi}$ to the largest positive integer such that $d(\beta) \cdot \beta \le \alpha$.

The elements of the set $\overline{\Pi}$ will be called *vertices* of the graph $D(\alpha)$.

These definitions imply that any root α is a linear combination of simple roots with coefficients $0, 1, 2$. We say that a vertex β of the graph $D(\alpha)$ is a *single vertex* if $2\beta \not\le \alpha$, and it is a *double vertex* if $2\beta \le \alpha$.

Also, the definitions show that, knowing the graph $D(\alpha)$, we can find the root α itself (as an element of the group $Z\Pi$).

§3. (RR, 1)-words

Any set of data determines the set of words and the set of RR-words (for letters we take the elements of the set Π). An RR-word will be called an (RR, i)*-word* if the greatest letter enters the word i times ($i = 1, 2$). In this section we study the structure of $(\mathrm{RR}, 1)$-words.

For brevity, we shall say "a graph" instead of "a graph of positive roots". The vertices of any graph are linearly ordered. The greatest vertex of a graph (the greatest letter of a word) will be called the *leading vertex* (*letter*). The greatest of the vertices of a graph that are not equal to the leading vertex (the greatest of the letters of a word that are not equal to the leading letter) will be called the *second vertex* (*letter*) *by priority.*

For a positive root α, the leading vertex of the graph $D(\alpha)$ will be denoted by $p_1(\alpha)$. If the root α is not simple, we denote by $p_2(\alpha)$ the second vertex of $D(\alpha)$ by priority.

For a positive root α, let $M^+(\alpha)$ denote the set of all simple roots β such that the element $\alpha - \beta$ is a root.

Remark 3.1. Definition 2.1 implies that the following vertices constitute the set $M^+(\alpha)$: all single vertices of $D(\alpha)$ that have exactly one adjacent single vertex; the double vertex of $D(\alpha)$ that has at most one adjacent double vertex (this vertex coincides with z only if z is the only double vertex of $D(\alpha)$).

If u is a word that is not a letter, we denote by $\widetilde{u}$ the word obtained from u by discarding the last letter.

Till the end of this section, we fix an $(\mathrm{RR}, 1)$-word u and denote by p_1, p_2, q its leading letter, the second letter by priority, and the last letter, respectively.

Lemma 3.1. *If an* $(\mathrm{RR}, 1)$*-word* u *is not a letter, then its last letter* q *is contained in* $M^+(|u|)$.

Proof. Under the assumptions of the lemma, the word u is regular, and the leading letter p_1 occurs in it only once. Hence, u begins with the only appearance of its leading letter p_1. Since the word u is not a letter, the word $\widetilde{u}$ begins with the only appearance of its leading letter p_1 as well. Therefore, the word $\widetilde{u}$ is regular, and its composition is rooted, because it is a regular subword of the RR-word u. Hence, the element $|\widetilde{u}| = |u| - |q|$ is a root. But this means exactly that $q \in M^+(|u|)$. $\square$

Lemma 3.2. *If an* $(\mathrm{RR}, 1)$*-word* u *is not a letter, then* $p_1 \notin [p_2, q]$.

Proof. Suppose u satisfies the assumptions of the lemma; we denote by w the ending of the word u that begins with the last occurrence of the letter p_2. The word w begins with the only occurrence of its leading letter p_2. This means that this word is regular. The composition of this word is rooted because it is a regular subword of the RR-word u. Therefore, the graph $D(|w|)$ is connected, and moreover, $p_2, q \in D(|w|)$. Hence, $[p_2, q] \subseteq D(|w|)$.

Let $p_1 \in [p_2, q]$; then $p_1 \in D(|w|)$. But the letter p_1 does not occur in the word w because this word is regular and p_2 is its first letter. We arrive at a contradiction. □

Lemma 3.3. *The last letter q of an* $(\mathrm{RR}, 1)$*-word u is uniquely determined by the following conditions*: *if u is not a letter, then $q \in M^+(|u|)$ and $p_1 \notin [p_2, q]$.* (This is an immediate consequence of Lemmas 3.1 and 3.2.)

Corollary 3.1. *Let α be a positive root such that the leading vertex of its graph is single. Then there exists only one* RR*-word the composition of which is α.*

Now we present some properties of $(\mathrm{RR}, 1)$-words to be used in the sequel.

Remark 3.2. Any nonempty beginning v' of an $(\mathrm{RR}, 1)$-word v starts with the only occurrence of the leading letter of v, and v' is an $(\mathrm{RR}, 1)$-word. Consequently, its composition is a root.

Indeed, let v be an $(\mathrm{RR}, 1)$-word. The word v is regular and the leading letter p_1 occurs in it only once. It follows that this word begins with the letter p_1. Therefore, any nonempty beginning v' of v starts with p_1. Hence, the word v' is regular, and v' is an RR-word because it is a regular subword of the RR-word v; consequently, the element $|v'|$ is a root.

Let α be a positive root, but not a simple root, and let the leading vertex p_1 of its graph be single; we denote by α^- the positive root that is uniquely determined by the following conditions: $|p_1| \le \alpha^- \le \alpha$; a vertex y of the graph $D(\alpha)$ is a vertex of the graph $D(\alpha^-)$ if and only if $p_1 \in [y, p_2)$, where p_2 is the second vertex of $D(\alpha)$ by priority; and any vertex y of $D(\alpha^-)$ has the same multiplicity in $D(\alpha^-)$ as in $D(\alpha)$.

§4. $(\mathrm{RR}, 2)$-WORDS

Let α be a positive root such that its graph has at least one double vertex. We introduce the following notation: $z_1(\alpha)$ is the only single vertex of $D(\alpha)$ that has only one adjacent vertex; $x_1^-(\alpha)$ is the only single vertex of $D(\alpha)$ that has an adjacent double vertex; $x_2^+(\alpha)$ is the only double vertex of $D(\alpha)$ that either has only one adjacent double vertex, or (if it is the only double vertex of $D(\alpha)$) coincides with z.

Throughout this section, u is an RR-word and α is the composition of u; next, p_1 is the leading letter of u, and u_1, u_2 are words such that $u = u_1u_2$ and the letter p_1 occurs in each of them exactly once.

We denote the greatest vertex of a graph D by $\max(D)$.

The words u_1, u_2 begin with the only occurrences of their leading letter p_1. Hence, they are regular.

Remark 4.1. It is well known that if v and w are regular words, then the word vw is regular if and only if $w < v$.

Corollary 4.1. 1) *u_1 and u_2 are* $(\mathrm{RR}, 1)$*-words* (*being regular subwords of the* RR*-word u with one occurrence of their leading letter*) *and, consequently, the composition of the words u_1, u_2 is rooted*;

2) *$u_2 < u_1$.*

For any two words v, w we write $w \prec v$ if $w < v$ and none of the words v, w is a beginning of the other. If $u_2 < u_1$, then there are two possibilities: either u_1 is a proper beginning of u_2, or $u_2 \prec u_1$. We start with analyzing the first of these cases.

Lemma 4.1. *If the word u_1 is a beginning of the word u_2, then*:
1) $|u_1| = [z, x_2^+(|u|)]$ *and* $|u_2| = [z, z_1(|u|)]$;
2) $\max(p_1, x_1^-(|u|)] > \max(p_1, z]$.

Proof. 1) Since u_1 is a beginning of u_2, we have $|u_1| \le |u_2|$. Hence, all vertices of the graph $D(|u|)$ are contained in $D(|u_2|)$, i.e., $[z_1(|u|), z] \le |u_2|$.

Suppose that the latter inequality is strict. Statement 1) of Corollary 4.1 shows that the elements u_1, u_2 are roots. Therefore, $2z \le |u_2|$.

Since u_1 is a proper beginning of u_2, there exists a letter q_2 such that $u_1 q_2$ is a beginning of u_2. By their choice, the words u_1, u_2 begin with the only occurrences of their leading letter p_1. Hence $q_2 < p_1$, which implies that $u_1 q_2$ is a regular word.

This fact and Remark 4.1 show that the word $u_1 u_1 q_2$ is regular. But the word $u_1 u_1 q_2$ is a subword of the RR-word u, so the composition of $u_1 u_1 q_2$ is rooted. Moreover, the letter p_1 occurs in it twice; hence, the letter z also occurs in it twice. Therefore, z occurs in u_1. So, $3z = z + 2z \le |u_1| + |u_2| \le |u|$, a contradiction.

2) Assume that q_2 is a double vertex of $D(|u|)$. Then the relation $|u_1| = [z, x_2^+(|u|)]$ (statement 1)) implies that the letter q_2 occurs in u_1. Moreover, the word $u_1 u_1 q_2$ is a beginning of u. Hence, q_2 occurs in u at least three times, which is impossible, because the composition of u is rooted. Thus, q_2 is a single vertex of the graph $D(|u|)$.

It has been shown above that the word $u_1 q_2$ is regular. Being a regular subword of the RR-word u, it is itself an RR-word. This implies, in particular, that its composition is rooted. Therefore, the graph $D(|u_1 q_2|)$ is connected.

Thus, $|u_1| = [z, x_2^+(|u|)]$ and q is a single vertex of the graph $D(|u|)$. Consequently, $q_2 = x_1^-(|u|)$ and $|u_1 q_2| = [z, x_1^-(|u|)]$, so that $u_1 q_2$ is an RR-word. This and Lemma 3.3 yield statement 2). □

Now we consider the case where $u_2 \prec u_1$. Then

$$u_1 = a q_1 b, \quad u_2 = a q_2 c, \quad q_1 > q_2, \tag{1}$$

where q_1, q_2 are letters and a, b, c are words (possibly, empty).

Lemma 4.2. *If u is an* $(\mathrm{RR}, 2)$*-word and $u_2 \prec u_1$, then the word a in* (1) *is not empty, and $u_1 a q_2$ is an* RR*-word such that the graph of its composition has double vertices and, consequently, z is a double vertex of $D(|u_1 a q_2|)$.*

Proof. If the word a is empty, then u_2 begins with the letter $q_2 < p_1$ (in fact, $q_2 < q_1 \le p_1$). But u_2 begins with p_1 by definition. Hence, the word a is not empty.

The word $a q_2$ is regular, because it begins with the only occurrence of its leading letter. Thus, $u_1 > a q_2$, so that the word $u_1 a q_2$ is regular. Being a regular subword of the RR-word u, the word $u_1 a q_2$ is an RR-word. In particular, this means that the composition of the word $u_1 a q_2$ is rooted. Hence, $2z \le |u_1 a q_2|$.

Since p_1 occurs in each of the words u_1 and a, we see that p_1 is a double vertex of the graph $D(|u_1 a q_2|)$. Thus, $D(|u_1 a q_2|)$ has double vertices. □

Lemma 4.3. *If u is an* $(\mathrm{RR}, 2)$*-word and $u_2 \prec u_1$, then the graph $D(|u_2|)$ has no double vertices.*

Proof. Suppose that $D(|u_2|)$ has double vertices. Then, necessarily, z is a double vertex of $D(|u_2|)$. It follows that $z \notin D(|a|)$: otherwise the letter z occurs at least three times in u. Hence, there are two occurrences of z in the word $q_2 c$. Therefore, z occurs in the word c. But then z occurs in $u_1 a q_2$ at most once, which contradicts Lemma 4.2. □

Lemma 4.4. *If u is an $(\mathrm{RR}, 2)$-word, $u_2 \prec u_1$, and a is as in (1), then all vertices of $D(|a|)$ are double vertices of $D(|u|)$, and $D(|a|) = (q_1, q_2)$, where $q_1 > \max(q_1, p_1)$, $q_1 > \max[q_2, p_1)$, and $\max(p_1, q_2] > \max(q_1, p_1)$.*

Proof. The words a, aq_1, aq_2 begin with the only occurrences of their leading letter p_1. Hence, they are regular. Being regular subwords of the RR-word u, they are RR-words as well.

In particular, their compositions are rooted. Therefore, the graphs $D(|a|)$, $D(|aq_1|)$, and $D(|aq_2|)$ are connected. But the decomposition (1) shows that all vertices of $D(|a|)$ are double vertices of $D(|u|)$. Hence, q_1 and q_2 are not vertices of $D(|a|)$ (otherwise, some letters would appear three times in u), but they have adjacent vertices in $D(|a|)$. Observe that q_1 and q_2 are distinct vertices.

The connected graph $D(|a|)$ has at most two adjacent vertices in the graph $D(|u|)$ (and these adjacent vertices are not in $D(|a|)$). Thus, $D(|a|) = (q_1, q_2)$. Moreover, p_1 is a single vertex of $D(|a|)$. This and Lemma 3.3 imply that $\max(p_1, q_2] > \max(q_1, p_1)$ (because aq_2 is an RR-word) and $\max(p_1, q_1] > \max(q_2, p_1)$ (because aq_1 is an RR-word). Recall that $q_1 > q_2$. Therefore, $\max(p_1, q_1] > \max(p_1, q_2] > \max(q_1, p_1)$. Hence, $q_1 > \max(q_1, p_1)$ and $q_1 > \max[q_2, p_1)$. □

Lemma 4.5. *If u is an $(\mathrm{RR}, 2)$-word and $u_2 \prec u_1$, then:*

1) *q_1 is the nearest vertex to p_1 in the interval $[z, p_1)$ such that $q_1 > \max(p_1, q_2]$;*
2) *the letter q_2 in (1) coincides with the letter $x_1^-(|u|)$;*
3) *$\max[z, p_1) > \max(p_1, x_1^-(|u|)]$.*

Proof. 2) Indeed, assume that q_2 is a double vertex of the graph $D(\alpha)$. Then $q_1 \not\leq |u_1|$, $q_1 \not\leq |a|$ by Lemma 4.3.

Hence, q_2 occurs in the word b (because $q_1 > q_2$). Then there is only one occurrence of q_2 in u_1. Let $\widetilde{u_1}$ be the beginning of u_1 that ends with the only occurrence of q_2. Then $\widetilde{u_1}$ is an RR-word, it has only one occurrence of its leading letter p_1, it ends with the only occurrence of q_2, and the letter q_1 occurs in it. Now, Lemma 3.3 shows that $\max(p_1, q_2]$ is greater than any vertex of $D(|\widetilde{u_1}|)$ not contained in the interval $[p_1, q_2]$.

Thus, $q_1 \notin (p_1, q_2]$ by Lemma 4.4. Hence, $\max(p_1, q_2] > q_1$, which contradicts Lemma 4.4.

Therefore, q_2 is a single vertex of the graph $D(|u|)$.

By Lemma 4.4, in the graph $D(|u|)$ the vertex q_2 has an adjacent double vertex. Thus, q_2 is a single vertex of $D(|u|)$, and in this graph it has an adjacent double vertex, i.e., $q_2 = x_1^-(|u|)$.

Statement 1) follows from statement 2) and Lemma 4.4, and statement 3) follows from statements 1) and 2). □

Lemma 4.6. *If u is an $(\mathrm{RR}, 2)$-word and $u_2 \prec u_1$, then the graph $D(|u_2|)$ coincides with the interval $(q_1, z_1(|u|)]$, where q_1 is the nearest vertex to p_1 in the interval $[z, p_1)$ such that $q_1 > \max(p_1, x_1^-(|u|)]$.*

Proof. By Lemma 4.3, no letter occurs in the word u_2 two or more times. Since the graph $D(|u_2|)$ is connected, we see that it is an interval.

Each vertex of $D(|u_1|)$ is double in $D(|u|)$. Indeed, otherwise $D(|u_1|)$ has a vertex t that is single in $D(|u|)$. Then $x_1^-(|u|) \leq [p_1, t] \leq |u_1|$, which is impossible. Thus, $[z_1(|u|), x_1^-(|u|)] \leq |u_2|$.

By Lemma 4.4, $|a| = (q_1, q_2)$. Moreover, $|a| \leq |u_2|$, because the word a is a beginning of u_2. Hence, $(q_1, q_2) \leq |u_2|$. Now, statement 2) of Lemma 4.5 shows that $(q_1, x_1^-(|u|)) \leq |u_2|$ and the inequality $[z_1(|u|), x_1^-(|u|)] \leq |u_2|$ yields $[z_1(|u|), q_1) \leq |u_2|$.

We prove that $|u_2| \le [z_1(|u|), q_1)$. Indeed, otherwise the inequality $[z_1(|u|), x_1^-(|u|)] \le |u_2|$ and the connectedness of the graph $D(|u_2|)$ would imply that the letter q_1 occurs in the word u_2. Since $D(|a|) = (q_1, q_2)$ (see Lemma 4.4), the letter q_1 does not occur in a. Hence, q_1 occurs in c (recall that $u_2 = aq_2c$, $q_1 > q_2$), and thus q_1 occurs in u_2 to the right of the only occurrence of q_2. Hence, q_1 occurs in the word u_1aq_2 at most once.

But the composition of the word u_1aq_2 is rooted, the letter p_1 occurs in it twice (see Lemma 4.2), and $q_1 \in [z, p_1)$ (see Lemma 4.5). Hence, the word u_1aq_2 has two occurrences of q_1, a contradiction. □

Let $\mathrm{Root}(2, \max 2)$ denote the set of all positive roots α such that the leading vertex p_1 of the graph $D(\alpha)$ is a double vertex and the vertex second by priority of the interval $[x_1^-(\alpha), z]$ is contained in the interval $(p_1, z]$. We denote by $\mathrm{Root}(2, \max 1)$ the set of all positive roots that do not belong to $\mathrm{Root}(2, \max 2)$, but the leading vertices of their graphs are double.

Lemma 4.7. *Let α be a positive root such that the leading vertex of its graph is double. Then there exists only one* RR-*word u the composition of which is equal to α. Namely, $u = u_1u_2$, where*

1) *u_2 is the only* $(\mathrm{RR}, 1)$-*word the composition of which is the sum of all vertices in the interval $[p_1, z_1(\alpha)]$ and all vertices y in the interval $[z, p_1)$ such that* $\max[y, p_1) < \max(p_1, x_1^-(\alpha)]$;

2) *u_1 is the only* $(\mathrm{RR}, 1)$-*word such that* $|u_1| = [x_2^+(\alpha), z]$ *if* $\alpha \in \mathrm{Root}(2, \max 1)$, *and* $|u_1| = [x_2^+(\alpha), z] + [m, z]$ *if* $\alpha \in \mathrm{Root}(2, \max 2)$, *where m is the nearest to the leading vertex p_1 of the graph $D(\alpha)$ among all vertices of the interval $(p_1, z]$ that are greater than all vertices of the interval $[x_1^-(\alpha), p_1)$.*

Proof. Let $u = u_1u_2$ be an RR-word the composition of which is equal to α (the words u_1 and u_2 were defined at the beginning of this section).

1) Lemmas 4.1, 4.6 imply that the composition of u_2 is the sum of all vertices in the interval $[p_1, z_1(|u|)]$ and all vertices y in the interval $[z, p_1)$ such that $\max[y, p_1) < \max(p_1, x_1^-(|u|)]$. By Corollary 3.1, the $(\mathrm{RR}, 1)$-word u_2 is uniquely determined by its composition.

2) By statement 1) of Corollary 4.1, u_1 is an $(\mathrm{RR}, 1)$-word. Since $u = u_1u_2$ and $|u| = \alpha$, we know the composition of the word u_1: $|u_1| = \alpha - |u_2|$, where the root $|u_2|$ is determined by property 1). By Corollary 3.1, the $(\mathrm{RR}, 1)$-word u_1 is uniquely determined by its composition. □

The following statement is an immediate consequence of Corollary 3.1 and Lemma 4.7.

Corollary 4.2. *For any positive root α, there exists exactly one* RR-*word the composition of which is equal to α.*

For a positive root α, we shall denote by $\mathrm{rrWord}(\alpha)$ the only RR-word the composition of which is equal to α.

§5. SR-words

For a regular associative word u, we introduce the following notation: $l(u)$ is the longest proper regular beginning of u; $r(u)$ is the longest proper regular ending of u; $\overline{l}(u)$ is the ending of u obtained from u by discarding the beginning $l(u)$; and $\overline{r}(u)$ is the beginning of u obtained from u by discarding the ending $r(u)$.

Let u, v be any two words; we shall write $v \boxed{\lhd} u$ if $u = wy_1u_1$ and $v = wy_2$ for some (possibly, empty) words w, u_1 and some letters y_1, y_2 such that $y_1 > y_2$. We recall a well-known property.

Remark 5.1. Any associative word can be written as a product $u = u_1 u_2 \cdots u_n$ of a nondecreasing sequence of regular words $u_1, u_2, \ldots, u_n$. The nonnegative integer n and the words $u_1 \leq u_2 \leq \cdots \leq u_n$ are determined uniquely by u.

The following property of regular words plays an essential role in this paper.

Lemma 5.1. *For any regular associative word a of length at least 2, there exist regular words b, c and a positive integer k such that $a = b^k c$ and $c \boxed{\prec} b$. Such a triple (b, c, k) is unique, and moreover, $b = l(a)$.*

Proof. Suppose that such a triple (b, c, k) exists; we prove that it is unique.

First, we show that $b = l(a)$. Assume the contrary. The word b is a regular proper beginning of a. Therefore, it is a regular beginning of $l(a)$. By assumption, b is a proper beginning of $l(a)$. Hence, either $l(a) = b^t b'$, where $1 \leq t < k$ and b' is a beginning of b, or $l(a) = b^k c'$, where c' is a proper beginning of c.

In the former case, we have $b \leq b'$, and by Remark 5.1 the word $b^t b'$ cannot be regular. But $l(a) = b^t b'$ is a regular word.

In the latter case, Remark 5.1 shows that $b > c'$. But we have assumed that $c \boxed{\prec} b$. The word c' is a proper beginning of c. Hence, c' is a beginning of b, whence $b \leq c'$.

Thus, both cases lead to a contradiction, which means that $b = l(a)$.

Next, we have $c \prec b$ because $c \boxed{\prec} b$. Hence, b is not a beginning of c. Therefore, k is the greatest integer such that b^k is a beginning of a. Thus, the integer k is determined uniquely.

The word c is obtained from a by discarding the beginning b^k. Therefore, the word c is also determined uniquely.

It remains to prove the existence of the triple (b, c, k). Let $b = l(a)$, and let k be the greatest integer such that b^k is a beginning of a; next, let c be the ending of a obtained by discarding the beginning b^k. It suffices to show that $c \boxed{\prec} b$.

The definition of regular words implies that $c < a$. But $a \leq b$, because b is a beginning of a. Hence, $c < b$.

By the definition of the integer k, the word b cannot be a beginning of c. Since $c < b$, we see that $c \prec b$, i.e., $b = w x_1 b_1$ and $c = w x_2 c_1$ for some (possibly, empty) words w, c_1, b_1 and some letters x_1, x_2 such that $x_1 > x_2$.

Assume that c_1 is not empty. Then $c'' \prec b$, where c'' is the word obtained from c by discarding the last letter. Hence, $c'' < b$. This and Remark 5.1 imply that the word $b^k c''$ is regular.

Since $b^k c''$ is the longest proper beginning of a (it is obtained from a by discarding the last letter), we see that $l(a) = b^k c''$ (by the definition of the word $l(a)$). But this is impossible, because $l(a) = b$ by construction, and the word c'' is not empty (the letter x_2 occurs in it). Thus, our assumption is false. Hence, $c \boxed{\prec} b$.

The lemma is proved. □

Now, let $\langle X, \Phi^+, \leq \rangle$ be a set of data for which the leading vertices of the graphs of positive roots are at most double (the set of data for B_n satisfies this condition). By Lemma 5.1, the set of SR-words for such a set of data splits into the disjoint union of classes $S(i, j, k)$ to be described below.

Let u be an SR-word, and let $u = a^{i(u)} b$ be its decomposition as in Lemma 5.1; here $i(u)$ is a positive integer and a, b are regular words such that $b \boxed{\prec} a$. We introduce additional parameters that characterize the word u; namely, let $j(u)$ and $k(u)$ be the numbers of occurrences of the leading letter of u in a and in b, respectively. For nonnegative integers i, j, k, for $S(i, j, k)$ we take the set of all SR-words u such that $i = i(u)$, $j = j(u)$, and $k = k(u)$.

Lemma 5.2. *Let $\langle X, \Phi^+, \leq \rangle$ be a set of data for which the leading vertices of the graphs of positive roots are at most double. Then the set of* SR*-words for this set of data is the union of the sets $S(i, j, k)$, where*
1) $1 \leq i \leq 3$;
2) $j \in \{1, 2\}$;
3) $0 \leq k \leq j$;
4) *if $i = 2$ and $j = 2$, then $k = 0$;*
5) *if $i = 3$, then $j = 1$ and $k = 0$.*

Proof. 1) Let $u = a^i b$ be the decomposition of an SR-word u, where a and b are regular words such that $b \mathbin{\boxed{<}} a$. Assume that $i \geq 3$. Then, by Remark 4.1, the word $a^{i-1}b$ is a regular proper subword of u. Hence, the composition of the word $a^{i-1}b$ is rooted, and consequently, the leading letter p_1 of u occurs in the word $a^{i-1}b$ at most twice.

But the word u is regular. Hence, it begins with the letter p_1. Therefore, p_1 appears in the word a at least once (by definition, the beginning a of the word u is not empty). Thus, $i - 1 \leq 2$, $i \leq 3$.

2) The word a is a regular proper subword of the SR-word u. Hence, the composition of a is rooted, and consequently, p_1 occurs in a at most twice, i.e., $j \leq 2$. Moreover, the leading letter p_1 of u appears at least once in a, i.e., $j > 0$.

3) The word b does not end with the letter p_1, because it is a regular word, and $b \mathbin{\boxed{<}} a$. Hence, any occurrence of p_1 in b is an occurrence of p_1 in the common beginning of a and b. Therefore, $0 \leq k \leq j$.

4) If $i = 2$, then the word ab is a regular proper subword of the SR-word u. Hence, the composition of ab is rooted, and consequently, the number $j + k$ of occurrences of p_1 in ab is not greater than 2. If $j = 2$, we see that $k = 0$.

5) If $i = 3$, then the word $a^2 b$ is a regular proper subword of the SR-word u. Hence, the composition of $a^2 b$ is rooted, and consequently, the number $2j + k$ of occurrence of p_1 in $a^2 b$ is not greater than 2. But $j > 0$, so that $j = 1$ and $k = 0$. □

We can easily list all triples (i, j, k) satisfying the conditions of Lemma 5.2: $(1, 1, 0)$, $(1, 1, 1)$, $(1, 2, 0)$, $(1, 2, 1)$, $(1, 2, 2)$, $(2, 1, 0)$, $(2, 1, 1)$, $(2, 2, 0)$, and $(3, 1, 0)$.

Corollary 5.1. *Let $\langle X, \Phi^+, \leq \rangle$ be a set of data for which the leading vertices of the graphs of positive roots are at most double (in particular, it can be the set of data for B_n). Then the set of* SR*-words for this set of data is the disjoint union of the sets $S(i, j, k)$, where (i, j, k) is one of the nine triples $(1, 1, 0)$, $(1, 1, 1)$, $(1, 2, 0)$, $(1, 2, 1)$, $(1, 2, 2)$, $(2, 1, 0)$, $(2, 1, 1)$, $(2, 2, 0)$, and $(3, 1, 0)$.*

In what follows, we shall study each of these nine sets $S(i, j, k)$. But before doing this, we discuss some useful properties of regular words. The first of these properties is well known.

Remark 5.2. Let y, z be regular subwords of a regular word x. If the intersection of these subwords is not empty, then their union is again a regular word.

Thus, by Lemma 5.1, for any regular word u the union of the subwords $l(u)$, $r(u)$ coincides with u. As a corollary, we obtain the following property of regular words.

Lemma 5.3. *Any regular subword of a regular word x is either a subword of $l(x)$, or a subword of $r(x)$.*

Lemma 5.3 implies the following characterization of RR-words and SR-words.

Remark 5.3. A word u is an RR-word if and only if it is regular, its composition is rooted, and the words $l(u)$ and $r(u)$ are RR-words.

Remark 5.4. A word u is an SR-word if and only if it is regular, its composition is not rooted, and the words $l(u)$ and $r(u)$ are RR-words.

§6. The set $S(1,1,0)$

Remark 6.1. The set $S(1,1,0)$ consists of all SR-words with only one occurrence of the leading letter.

For a word v, we shall denote by $p_1(v)$ its leading letter, by $p_2(v)$ its letter second by priority, and by $\widetilde{v}$ the word obtained from v by discarding the last letter.

In what follows, u will be an $(\mathrm{SR},1)$-word and q will be the last letter of u.

First, we describe the words of length 2 belonging to $S(1,1,0)$. We denote the set of such words by $S(len{=}2)$.

Remark 6.2. The set $S(len{=}2)$ consists of all words xy such that the letters x, y are not adjacent in the graph B_n and $x > y$.

We split the set of $(\mathrm{SR},1)$-words of length exceeding 2 into the disjoint union of three subsets: the set $S(q{>}p_2)$ of words the last letter of which is greater than $p_2(\widetilde{u})$; the set $S(q{=}p_2)$ of words the last letter of which is equal to $p_2(\widetilde{u})$; and the set $S(q{<}p_2)$ of words the last letter of which is smaller than $p_2(\widetilde{u})$. It is obvious that if $u \in S(q{=}p_2)$ or $u \in S(q{>}p_2)$, then $r(u) = q$. Using Remark 5.4, we arive at the following description of the sets $S(q{>}p_2)$ and $S(q{=}p_2)$.

Remark 6.3. The words u in the set $S(q{>}p_2)$ are in one-to-one correspondence with the pairs (α, q) in which α is a nonsimple positive root such that the leading vertex p_1 of its graph is single, and q is a letter not adjacent to any vertex of the graph $D(\alpha)$ and such that $p_2(\alpha) < q < p_1$. This correspondence is given by the formula $u = \mathrm{red}(\alpha) \cdot q$, where $\mathrm{red}(\alpha)$ stands for the reduced word of composition α.

Remark 6.4. The words u in the set $S(q{=}p_2)$ are in one-to-one correspondence with the nonsimple positive roots α such that the leading vertex p_1 of the graph $D(\alpha)$ is single and the element $\alpha + |p_2(\alpha)|$ is not a root. This correspondence is given by the formula $u = \mathrm{red}(\alpha) \cdot p_2(\alpha)$.

The set $S(q{<}p_2)$ can be split into the disjoint union of two subsets: the set $S(p_2 1)$ of words in $S(q{<}p_2)$ with one occurrence of the letter second by priority and the set $S(p_2 2)$ of words in $S(q{<}p_2)$ with two occurrence of the letter second by priority.

Lemma 6.1. *The words u in $S(p_2 1)$ are in one-to-one correspondence with the nonsimple positive roots α such that the leading vertex p_1 and the vertex p_2 second by priority in the graph $D(\alpha)$ are single and nonadjacent, and the vertex q of the interval (p_2, p_1) that is adjacent to p_2 is greater than any vertex t of the graph $D(\alpha)$ such that $p_2 \in (t, q)$. This correspondence is given by the formula $u = \mathrm{red}(\alpha) \cdot q$.*

Proof. Let $u \in S(p_2 1)$. We denote by w the ending of u that begins with the only occurrence of the letter p_2. The word w is regular and, being a regular proper subword of the SR-word u, it is an RR-word. Similarly, the words $\widetilde{u}$ and $\widetilde{w}$ obtained from u and v by discarding their last letter q are RR-words.

Let α denote the composition of the word $\widetilde{u}$. Since $\widetilde{u}$, $\widetilde{w}$, and w are RR-words, the elements α, $|\widetilde{w}|$, and $|w| = |\widetilde{w}| + |q|$ are positive roots.

Since p_1 occurs in the RR-word $\widetilde{u}$ only once, this word is an $(\mathrm{RR},1)$-word. Moreover, $2|p_2| \nleq \alpha$. Therefore, applying Lemma 3.3 to $\widetilde{u}$, we can find the composition of $\widetilde{w}$: the graph $D(|\widetilde{w}|)$ consists of all vertices t of $D(\alpha)$ such that $p_2 \in [t, p_1)$, and the multiplicities of these vertices in the graphs $D(|\widetilde{w}|)$ and $D(\alpha)$ are equal. The element $|w| = |\widetilde{w}| + |q|$ is a root, and the graph $D(\alpha)$ has at least two single vertices (say, p_1, p_2). Hence, $q \in (p_1, p_2)$, and consequently, the vertices p_1 and p_2 are not adjacent.

Since $q \in (t, p_2)$ for every letter $t \neq p_2$ occurring in the word $\widetilde{w}$, and the $(\mathrm{RR}, 1)$-word w ends with the letter q, Lemma 3.3 shows that q is the second by priority letter of w. Now, the description of the root $|\widetilde{w}|$ implies that q is greater than any vertex t of the graph $D(\alpha)$ such that $p_2 \in (t, q)$.

The converse statement of the lemma can easily be obtained from Remark 5.4 and Lemma 3.3. □

Lemma 6.2. *The words u in $S(p_2 2)$ are in one-to-one correspondence with the nonsimple positive roots α such that the graph $D(\alpha)$ has at least two double vertices, the leading vertex p_1 of this graph is single, and the vertex p_2 second by priority is double, distinct from z, and has exactly one adjacent double vertex. This correspondence is given by the formula $u = \mathrm{red}(\alpha) \cdot q_1$, where q_1 stands for the double vertex of the graph $D(\alpha)$ adjacent to p_2.*

Proof. Let $u \in S(p_2 2)$. Since u is an SR-word, the proper ending w of this word that begins with the right occurrence of the letter p_2 is again an RR-word; consequently, its composition is rooted.

Since $u \in S(p_2 2)$, the last letter q of u is smaller than p_2. Hence, the word $\widetilde{w}$ begins with p_2, and it is the only occurrence of this letter in $\widetilde{w}$. It follows that the composition of $\widetilde{w}$ is rooted, because it is a regular subword of the RR-word w.

Applying Lemma 3.3 to the RR-word $\widetilde{u}$, we see that $|\widetilde{w}| = [p_2, x_2^+]$. Moreover, above it was shown that the elements $|\widetilde{w}|$ and $|w| = |\widetilde{w}| + |q|$ are roots (and consequently, the graph $D(|w|)$ is connected), and the element $|u|$ is not a root, because u is an SR-word. Hence, $p_2 \in (q, p_1)$, and the vertex q is adjacent to p_2.

Moreover, the vertex p_2 of the graph $D(\alpha)$ is double. Therefore, q is also a double vertex of the graph $D(\alpha)$, and so the graph $D(\alpha)$ has at least two double vertices. Now it is clear that $q_1 = q$ and $|\widetilde{w}| = [q, x_2^+]$.

Assume that $p_2 \neq x_2^+$, i.e., that the vertex p_2 has two adjacent double vertices in the graph $D(\alpha)$. Let t denote the vertex that is adjacent to p_2 and is distinct from q. Since, by its choice, the $(\mathrm{RR}, 1)$-word w ends with the letter q, the letter t is the second letter of the word w, whence $|\widetilde{w}| = [q, x_2^+]$ and $t < q$.

We denote by w' the longest subword of $\widetilde{u}$ that begins with the left occurrence of p_2 and has no other occurrences of p_2. Then w' is an $(\mathrm{RR}, 1)$-word in which q is the second letter. Moreover, $t < q$ and t is the second letter of w. Hence, $w' > w$. Therefore, $w'w$ is a regular word and $w'w$ is a proper ending of the SR-word u. So, the element $|w'w|$ is a root.

But q is a double vertex of $D(\alpha)$. Hence, the letter q occurs in w' two times. Therefore, q occurs in $w'w$ three times, in contradiction with the fact that the element $|w'w|$ is a root.

Thus, $p_2 = x_2^+$. But we have already proved that the graph $D(\alpha)$ has at least two double vertices. Hence, $p_2 \neq z$.

The converse statement of the lemma can easily be obtained from Remark 5.4 and Lemma 3.3. □

Remark 6.5. The set $S(1, 1, 0)$ is the disjoint union of the five subsets $S(len{=}2)$, $S(q{>}p_2)$, $S(q{=}p_2)$, $S(p_2 1)$, and $S(p_2 2)$, which are described in Remarks 6.2, 6.3, 6.4 and Lemmas 6.1, 6.2.

§7. The set $S(1,1,1)$

Let u be a word in $S(1,1,1)$. Recall that u can be represented in the form $u = vw$, where v, w are regular words with the following properties: their graphs have the same leading vertex p_1, this vertex is a single vertex of both graphs, and $w \boxdot v$.

The words v and w are proper regular subwords of the SR-word u; hence, v and w are RR-words. We denote by $P(1,1)$ the set of pairs (v,w) of RR-words such that the word vw is regular, the graphs $D(|v|)$ and $D(|w|)$ have the same leading vertex p_1, it is a single vertex of both graphs, and $w \boxdot v$.

Remark 7.1. The words u in $S(1,1,1)$ are in one-to-one correspondence with the pairs (v,w) in $P(1,1)$ such that the element $|v| + |w|$ is not a root. This correspondence $h : P(1,1) \to S(1,1,1)$ is given by the formula $h(u,v) = vw$.

Remark 7.2. Definition 2.1 implies that if α and β are positive roots in the set of data for B_n such that $\beta < \alpha$, then:

1) either $\alpha - \beta$ is a simple root, or there is a positive root γ such that $\beta < \gamma < \alpha$;

2) if the root β is not simple and q is an element of the set $M^+(\alpha)$ such that its multiplicities in the graphs $D(\beta)$ and $D(\alpha)$ are equal, then $q \in M^+(\beta)$.

Lemma 7.1. *Let α and β be positive roots such that the leading vertex p_1 of the graph $D(\alpha)$ is single and $|p_1| \le \beta < \alpha$. Then* $\mathrm{rrWord}(\beta) > \mathrm{rrWord}(\alpha)$.

Proof. Statement 1) of Remark 7.2 allows us to assume that the element $\alpha - \beta$ is a simple root. Let t denote the letter such that $|t| = \alpha - \beta$, and let u, v be the only RR-words the compositions of which are equal to α and β, respectively.

Assume that the statement of the lemma fails. Consider a counterexample in which the root α is minimal.

First, we assume that u ends with the letter t. Then the word $\widetilde{u}$ obtained from u by discarding the last letter has the same composition as v. By Corollary 3.1, it follows that $\widetilde{u} = v$. Hence, $u = vt < v$.

Thus, the last letter q of u is not t. Let $D^*(\alpha)$ and $D^*(\beta)$ be the graphs obtained (respectively) from $D(\alpha)$ and $D(\beta)$ by discarding the vertex p_1. Further, let $D^+(\alpha)$ and $D^+(\beta)$ be the connected components of the graphs $D^*(\alpha)$ and $D^*(\beta)$ that contain their leading vertices $p_2(\alpha)$ and $p_2(\beta)$. By Lemma 3.3, we have $q \in M^+(\alpha)$ and $q \in D^+(\alpha)$.

Since $|t| = \alpha - \beta$, we see that $t \in M^+(\alpha)$. Moreover, $t \ne q$, $q \in M^+(\alpha)$, and the leading vertex p_1 of $D(\alpha)$ is single. Hence, $p_1 \in [t,q]$. But $q \in D^+(\alpha)$, whence $t \notin D^+(\alpha)$. It follows that $p_2(\beta) = p_2(\alpha)$, $D^+(\beta) = D^+(\alpha)$. Therefore, $t \notin D^+(\beta)$. Since $p_1 \in [t,q]$, we conclude that $q \in D^+(\beta)$.

The multiplicities of the vertex q in the graphs $D(\alpha)$ and $D(\beta)$ are equal, because $|t| = \alpha - \beta$, $p_1 \in [t,q]$. Since, $q \in M^+(\alpha)$, statement 2) of Remark 7.2 shows $q \in M^+(\beta)$. Next, since $q \in D^+(\beta)$, with the help of Lemma 3.3 wee see that the word v ends with the letter q, i.e., $v = \widetilde{v}q$, where $\widetilde{v}$ is the word obtained from v by discarding the last letter.

The word u is regular and it is not a letter. Hence, it begins with its leading letter p_1, and its last letter q is smaller than p_1. In particular, $q \ne p_1$.

The word v is regular and p_1 occurs in it only once. Hence, v begins with the only occurrence of its leading letter p_1. Therefore, the relations $v = \widetilde{v}q$ and $q \ne p_1$ imply that $\widetilde{v}$ begins with the the only occurrence of its leading letter p_1; consequently, the word $\widetilde{v}$ is not empty, it is regular, and $|p_1| \le |\widetilde{v}|$.

Being a regular subword of the RR-word v, the word $\widetilde{v}$ is itself an RR-word, and consequently, the element $|\widetilde{v}|$ is a root. Since $q \in M^+(\alpha)$ and $u = \widetilde{u}q$, the element $|\widetilde{u}|$ is a root.

It has been proved above that the requirements of the lemma are fulfilled for the pair of roots $\alpha - |q|$, $\beta - |q|$. Hence, $\widetilde{u} < \widetilde{v}$ by the choice of the root α. It follows that $u = \widetilde{u}q < \widetilde{v}q = v$, which contradicts the choice of α. This contradiction proves the lemma. $\square$

For a word w and a letter q, denote by $d_q(w)$ the number of occurrences of q in w.

Lemma 7.2. *If a pair* (v, w) *belongs to* $P(1,1)$, *then the element* $|v| + |q|$ *is a root* (*here* q *is the last letter of* w).

Proof. Since $(v, w) \in P(1,1)$, w is an RR-word, and consequently, the element $|w|$ is a root. Hence, the letter q occurs in w at most twice.

If $d_q(v) = 0$, then q is not a vertex of the graph $D(|v|)$. But $|v|$ is a root. Hence, the element $|v| + |q|$ is a root. Thus, we may assume that $d_q(v) \geq 1$.

Since $w \boxed{\lhd} v$, we conclude that $|w| \leq |v| + |q|$. Therefore, if $d_q(w) \leq d_q(v)$, then $|w| \leq |v|$. Then $w \geq v$ by Lemma 7.1. But the words v, w, and vw are regular (because $(v, w) \in P(1,1)$), and $v > w$ by Remark 4.1. We arrive at a contradiction. Thus, $d_q(w) > d_q(v)$. Moreover, $d_q(w) \leq d_q(v) + 1$ because $|w| \leq |v| + |q|$. Hence, $d_q(w) = d_q(v) + 1$. Using the inequalities $d_q(w) \leq 2$ and $d_q(v) \geq 1$, we see that $d_q(w) = 2$ and $d_q(v) = 1$.

Since $(v, w) \in P(1,1)$, the word w is an $(\mathrm{RR}, 1)$-word, and its length is at least two. Hence, its beginning $\widetilde{w}$ obtained by discarding the last letter is not empty. Therefore, by Remark 3.2, $\widetilde{w}$ is an RR-word and $|p_1| \leq |\widetilde{w}|$, so that the element $|\widetilde{w}|$ is a root.

Now from the relation $d_q(w) = 2$ we conclude that q is a single vertex of the graph $D(|\widetilde{w}|)$, i.e., $q \in M^+(|\widetilde{w}|)$. Hence, in the graph $D(|\widetilde{w}|)$ the vertex q has an adjacent single vertex, and all vertices in the interval $(q, z]$ are double vertices of this graph (if $q = z$, we agree that this interval is empty).

Next, $|\widetilde{w}| \leq |v|$ because $|w| \leq |v| + |q|$. Therefore, all vertices of the interval $(q, z]$ are double vertices of the graph $D(|v|)$.

All double vertices of the graph $D(|v|)$ are contained in the interval $(q, z]$. Indeed, otherwise q would be a double vertex of $D(|v|)$ (because the set of double vertices of a root is an interval), and this contradicts the relation $d_q(v) = 1$. But $|\widetilde{w}| \leq |v|$ and all vertices of the interval $(q, z]$ are double vertices of $D(|\widetilde{w}|)$. Therefore, all double vertices of $D(|v|)$ form the interval $(q, z]$.

Also, $|\widetilde{w}| \leq |v|$, and in the graph $D(|\widetilde{w}|)$ the vertex q has an adjacent single vertex. Therefore, q has an adjacent single vertex in the graph $D(|v|)$.

Thus, the set of all double vertices of $D(|v|)$ is the interval $(q, z]$, and q is a single vertex of this graph and has an adjacent single vertex in it. Therefore, $q \in M^+(|v|)$, i.e., the element $|v| + |q|$ is a root. $\square$

If (v, w) is a pair in $P(1,1)$, then $w \boxed{\lhd} v$. Hence, the length of the word v is not smaller than that of w. Let $\widehat{v}$ denote the beginning of v such that its length is equal to the length of w; we denote by $\widehat{q}$ the last letter of $\widehat{v}$. The relation $w \boxed{\lhd} v$ implies that $\widehat{v} = \widetilde{w}\widehat{q}$, $w = \widetilde{w}q$, and $q < \widehat{q}$.

Lemma 7.3. *Let* (v, w) *be a pair in* $P(1,1)$. *Then*:

0) *if* v' *is a beginning of* v *the length of which is not smaller than the length of* w, *then* $(v', w) \in P(1,1)$;

1) *the last letter* q *of* w *is smaller than the letter* $p_2(v')$ *of* v' *second by priority*;

2) $p_2(v') = p_2'$, *where* p_2' *is the vertex of the graph* $D(|v'| + |q|)$ *second by priority*;

3) q *cannot be the last letter of the word* $\mathrm{rrWord}(|v'| + |q|)$;

4) $p_1 \in (p_2', q)$.

Proof. 0) By Remark 3.2, v' is an $(\mathrm{RR},1)$-word.

Since v' is a beginning of v and the length of v' is not smaller than that of w, the relation $w \boxed{<} v$ implies that $w \boxed{<} v'$. Hence, $(v',w) \in P(1,1)$.

Because of property 0), it suffices to prove properties 1)–4) in the case where $v' = v$. Below, p_2 stands for the vertex of $D(|v|+|q|)$ second by priority.

1) Since $(v,w) \in P(1,1)$, the word w is regular, its length is at least two, and the letter p_1 appears in it only once. Hence, the last letter q of w is smaller than its leading letter p_1.

Assume that statement 1) of the lemma fails. Then $q = p_2(v)$, and the inequality $q < \widehat{q}$ shows that $\widehat{q} = p_1$.

On the other hand, the beginning $\widehat{v}$ of v starts with the only occurrence of its leading letter p_1. Therefore, the word $\widehat{v}$ is regular. Hence, it cannot end with its leading letter, i.e., $\widehat{q} < p_1$. This contradiction proves property 1).

Property 2) follows from 1).

3) By Lemma 7.2, the element $\alpha = |v| + |q|$ is a root, i.e., $q \in M^+(\alpha)$.

Assume that statement 3) of the lemma fails. Since $q < p_1$ and $\alpha = |v|+|q|$, the letter p_1 appears in the regular word $u = \mathrm{rrWord}(\alpha)$ only once. Hence, this word begins with its leading letter p_1. Therefore, the word $\widetilde{u}$ obtained from u by discarding the last letter is regular and, being a regular subword of the RR-word u, it is itself an RR-word. But, by Corollary 3.1, the RR-word $\widetilde{u}$ is determined by its composition. Therefore, $\widetilde{u} = v$ because $|\widetilde{u}| = \alpha - |q| = |v|$. Hence, $u = vq$. Now, the relation $w \boxed{<} v$ implies that $w \boxed{<} u$, whence $w < u$.

On the other hand, $|p_1| \le |w| \le |v| + |q| = \alpha$ and, by Lemma 7.1, $w \ge u$. This contradiction proves property 3).

4) Assume that statement 4) of the lemma fails, i.e., $p_1 \notin (p_2, q)$. Then the relations $p_1 \ne p_2$ and $p_1 \ne q$ imply $p_1 \notin [p_2, q]$. But $q \in M^+(\alpha)$ and $q < p_1$. Therefore, by Lemma 3.3, the word u ends with the letter q, and this contradicts property 3) proved above. □

Statement 4) of Lemma 7.3 implies the following claim.

Corollary 7.1. *If (v,w) is a pair in $P(1,1)$, then the leading vertex p_1 of the graph $D(|v|+|q|)$ has two adjacent vertices in this graph.*

We denote by Root_1^* the set of all positive roots such that the leading vertices of their graphs are single and have two adjacent vertices.

Lemma 7.4. *Let (v,w) be a pair in $P(1,1)$. Then:*

1) *the vertex q has the same multiplicity in the graph $D(\alpha)$ as in the graph $D(|w|)$;*

2) *$|q| \le \alpha^- \le |w|$, where $\alpha = |v| + |q|$;*

3) *if a vertex of the graph $D(\alpha)$ is distinct from its leading vertex p_1 and is greater than all vertices of the graph $D(\alpha^-)$ except p_1, then this vertex is not a vertex of the graph $D(|w|)$;*

4) *$\widehat{q} = m$, where m is the nearest to p_1 among all vertices of the graph $D(\alpha)$ (where $\alpha = |v| + |q|$) that are greater than all vertices of the graph $D(\alpha^-)$ except p_1;*

5) *the vertex m has an adjacent vertex in the graph $D(|w|)$.*

Proof. 1) Since $|w| \le |v|+|q|$, the multiplicity $d_q(\alpha)$ of the vertex q in the graph $D(\alpha)$ is not smaller than the multiplicity $d_q(|w|)$ of q in the graph $D(|w|)$. If statement 1) fails, then $d_q(|w|) < d_q(\alpha)$, in contradiction with property 3) of Lemma 7.3.

2) By statement 4) of Lemma 7.3, we have $p_1 \in (p_2, q)$. Hence, q is a vertex of the graph $D(\alpha^-)$, i.e., $|q| \le \alpha^-$. Next, $q \in M^+(|w|)$ by Lemma 3.1. Using property 1) and the relation $p_1 \in (p_2, q)$, we see that $\alpha^- \le |w|$.

3) Assume that statement 3) fails. Then the definition of the root α^- and property 2) show that $p_1 \in (p_2(|w|), q)$. By Lemma 3.3, the (RR, 1)-word w cannot end with the letter q, in contradiction with the choice of that letter. This contradiction proves statement 3).

4) By Lemma 7.3, the pair $(\widehat{v}, w)$ is an element of the set $P(1,1)$. Hence, by Lemma 7.2, the element $|\widehat{v}| + |q|$ is a root; we denote it by $\widehat{\alpha}$.

First, we show that $\widehat{q}$ is the letter of $\widehat{v}$ second by priority. Indeed, assume the contrary. Then $\widehat{q} < p_1$, because the word $\widehat{v}$ begins with the only occurrence of p_1 and the length of this word is at least two. Now from our assumption it follows that $\widehat{q} < p_2(\widehat{v})$.

By statement 1) of Lemma 7.3, we have $q < p_2(\widehat{v})$. The two inequalities above imply that $p_2(\widehat{\alpha}) = p_2(\widehat{v}) = p_2(w) = p_2(\widetilde{w})$. For brevity, we denote this letter by p_2.

The word w ends with the letter q, and the word $\widehat{v}$ ends with the letter $\widehat{q}$. Therefore, by Lemma 3.3, we have $p_1 \notin [p_2, q]$ and $p_1 \notin [p_2, \widehat{q}]$. But q and $\widehat{q}$ are elements of the set $M^+(\widehat{\alpha})$. Therefore, the vertices q and $\widehat{q}$ coincide.

But $q < \widehat{q}$ because $w \boxtimes v$, a contradiction. Thus, $\widehat{q}$ is the letter second by priority in the word $\widehat{v}$. Moreover, $q < \widehat{q}$ and $|w| < |\widehat{v}| + |q|$. Hence, $\widehat{q}$ is greater than all vertices of the graph $D(|w|)$ except p_1, i.e., $\widehat{q} > p_2(|w|)$.

Using property 2) proved above, we conclude that $\alpha^- \le |w|$. Hence, $p_2(\alpha^-) \le p_2(|w|)$. Thus, $\widehat{q} > p_2(\alpha^-)$.

Since $\widehat{v}$ is an RR-word, the element $|\widehat{v}|$ is a root. Hence, the graph $D(|\widehat{v}|)$ is connected. But $\widehat{v} = \widetilde{w} \cdot \widehat{q}$, and so $\widehat{q}$ has an adjacent vertex in the graph $D(|\widetilde{w}|)$. Therefore, $\widehat{q}$ has an adjacent vertex in the graph $D(|w|)$.

Assume that statement 4) fails, i.e., $\widehat{q} \ne m$. The inequality $\widehat{q} > p_2(\alpha^-)$ has been proved above; since $\alpha^- \le |w|$, we obtain $m \in (\widehat{q}, p_1)$. But $\widehat{q}$ has an adjacent vertex in the graph $D(|w|)$. Hence, m is a vertex of the graph $D(|w|)$, and this contradicts statement 3), which was proved before. Thus, we obtain statement 4).

5) Since $\widehat{q}$ has an adjacent vertex in $D(|w|)$, statement 4) implies statement 5). □

Lemma 7.5. *The pairs (v, w) in $P(1,1)$ are in one-to-one correspondence with the positive roots α such that the leading vertex p_1 of the graph $D(\alpha)$ is single and has two adjacent vertices. The correspondence $\mu : \mathrm{Root}_1^* \to P(1,1)$ is given by the formula $\mu(\alpha) = (\mathrm{rrWord}(\alpha - |q|), \mathrm{rrWord}(\beta))$, where q is the only element of $M^+(\alpha)$ that has the following properties: $p_1 \in (p_2, q)$, where p_2 is the vertex of the graph $D(\alpha)$ second by priority; $\beta = \alpha^- + (p_1, m)$ is a positive root (as before, m is the nearest to p_1 among all vertices of the graph $D(\alpha)$ that are greater than all vertices of the graph $D(\alpha^-)$ except p_1).*

Proof. Let Root_1 denote the set of positive roots such that the leading vertices of their graphs are single. If (v, w) is a pair in $P(1,1)$, then, by Lemma 7.2 and Corollary 7.1, the root $\alpha = |v| + |q|$ is an element of Root_1^*. Let $\nu : P(1,1) \to \mathrm{Root}_1^*$ be the mapping defined by the formula $\nu(v, w) = |v| + |q|$ $((v, w) \in P(1,1))$. We fix a root α in Root_1^*.

We show that the mapping ν is injective. The element q belongs to $M^+(\alpha)$ because $\alpha = |v| + |q|$. Also, $p_1 \in (p_2, q)$ by statement 4) of Lemma 7.3. The conditions $q \in M^+(\alpha)$ and $p_1 \in (p_2, q)$ determine the vertex q uniquely.

Therefore, the composition of the word v is determined uniquely; namely, $|v| = \alpha - |q|$. But v is an (RR, 1)-word; therefore, by Corollary 3.1, the word v is also uniquely determined.

Statements 1), 3), and 5) of Lemma 7.4 imply the following properties of the root $|w|$:

a) the multiplicity of the vertex q in the graph $D(\alpha)$ is the same as in the graph $D(|w|)$;

b) m is not a vertex of $D(|w|)$, but it has an adjacent vertex belonging to that graph.

Since $q \in M^+(\alpha)$, conditions a) and b) determine the root $|w|$ uniquely; namely, $|w| = \alpha^- + (p_1, m)$. But w is an $(\mathrm{RR}, 1)$-word, and by Corollary 3.1 the word w is determined uniquely. Thus, the words v and w are uniquely determined by the image $\nu(v, w)$, which means that the mapping ν is injective.

Now we show that the mapping ν is surjective. For this, consider the RR-words $v = \mathrm{rrWord}(\alpha - |q|)$ and $w = \mathrm{rrWord}(\beta)$. Let a be the only RR-word the composition of which is equal to $\beta - |q|$. Since the compositions of the $(\mathrm{RR}, 1)$-words a and $\widetilde{w}$ are equal, these words themselves are equal by Corollary 3.1. Hence, $w = a \cdot q$.

Under the assumptions of the lemma, we have $|q| \leq \alpha^-$. Therefore, $q < m$. By Lemma 3.3, the word $a \cdot m$ is a beginning of the word v. Since $w = a \cdot q$ and $q < m$, we see that $w \boxed{\lhd} v$, which implies that $(v, w) \in P(1, 1)$. Moreover, $\nu(v, w) = \alpha$. Hence, the mapping ν is surjective. But we have already seen that ν is injective; thus the mapping ν is bijective. It is obvious that the mapping μ is inverse to ν. □

Lemma 7.6. *Let (v, w) be a pair in $P(1, 1)$; then the element $|v| + |w|$ is a root if and only if the following conditions are satisfied*:

1) *all vertices of the graph $D(\alpha^-)$ are single* (*here $\alpha = |v| + |q|$ and q is the last letter of w*);

2) *z is a vertex of the graph $D(\alpha)$*;

3) $p_2 \in (p_1, z]$;

4) *if m is the vertex nearest to p_1 among all vertices of the interval $(p_1, z]$ that are greater than all vertices of the interval $[q, p_1)$, then m is double and belongs to $M^+(\alpha)$.*

Proof. We take a pair (v, w) in $P(1, 1)$ such that the element $|v|+|w|$ is a root, and prove properties 1)–4). By Lemma 7.2, the element $\alpha = |v| + |q|$ is a root.

1) Assume the contrary. Let y be a double vertex of the graph $D(\alpha^-)$. Then $2 \cdot |y| \leq \alpha^-$. But $\alpha^- \leq |w|$ by Lemma 7.5. Hence, $2|y| \leq |w|$. Now from the inequalities $2 \cdot |y| \leq \alpha^- \leq \alpha = |v| + |q|$ it follows that $|y| \leq |v|$. Therefore, $3 \cdot |y| \leq |v| + |w|$. But the element $|v| + |w|$ is a root, and so the latter inequality is impossible. This contradiction proves property 1).

2) Since $|p_1| \leq |v|$ and $|p_1| \leq |w|$, we have $2|p_1| \leq |v| + |w|$. Hence, the graph $D(|v| + |w|)$ has double vertices. Therefore, z is a double vertex of this graph. It follows that $|z| \leq |v| + |w|$. Moreover, $|v| + |w| \leq 2\alpha$ because $|v| \leq \alpha$ and $|w| \leq \alpha$. Hence, $|z| \leq 2\alpha$. Consequently, $|z| \leq \alpha$.

3) Assume the contrary. Then $p_1 \in (p_2, z]$, and by Lemma 7.5 properties 1) and 2) imply that all vertices of the graph $D(\alpha)$ are single, $q = z$, $|v| = \alpha - |z|$, and $|w| = (m, z]$ for a vertex m distinct from z.

Hence, z is a single vertex of the graph $D(|v| + |w|)$. But the element $|v| + |w|$ is a root whose graph $D(|v| + |w|)$ has double vertices (for example, p_1). This contradiction proves property 3).

4) By Lemma 7.5, from property 3) it follows that q is the only vertex of $D(\alpha)$ that has only one adjacent vertex and is such that $p_1 \in (q, z)$. Therefore $|w| = [q, m)$ by Lemma 7.5.

The vertex p_1 is double in the graph $D(|v| + |w|)$. Moreover, $m \in (p_1, z]$ and the element $|v| + |w|$ is a root. Hence, m is a double vertex of the graph $D(|v| + |w|)$. But the identity $|w| = [q, m)$ shows that m is not a vertex of the graph $D(|w|)$. Therefore, $2|m| \leq |v|$. Hence, m is a double vertex of the graph $D(\alpha)$.

If $m \notin M^+(\alpha)$ and m' is a vertex of the interval $[q, m]$ adjacent to m, then $2|m'| \leq |v|$. Also, $|m'| \leq |w|$ because $|w| = [q, m)$. It follows that $3|m'| \leq |v| + |w|$. But the element $|v| + |w|$ is a root. This contradiction proves property 4).

We prove the converse statement of the lemma. Let (v, w) be a pair belonging to $P(1, 1)$ and satisfying conditions 1)–4); we show that the element $|v| + |w|$ is a root.

Indeed, by Lemma 7.5 and condition 3), q is the only vertex of $D(\alpha)$ that has only one adjacent vertex and is such that $p_1 \in (q, z)$. By Lemma 7.5, we have $|w| = [q, m)$. But $|v| = \alpha - |q|$, and the last two identities and property 4) show that the element $|v| + |w|$ coincides with the root $[q, z] + (q, z]$. □

Remark 7.1 and Lemmas 7.5, 7.6 imply the following statement.

Corollary 7.2. *The words in $S(1,1,1)$ are in one-to-one correspondence with the positive roots α such that*

1) *the leading vertex p_1 of the graph $D(\alpha)$ is single and has two adjacent vertices in this graph*;

2) *at least one of the conditions of Lemma* 7.6 *fails for the element $\alpha - |q| + \beta$. Here q is the only element of $M^+(\alpha)$ that has the following properties*: $p_1 \in (p_2, q)$ (*where p_2 is the vertex of the graph $D(\alpha)$ second by priority*), *and $\beta = \alpha^- + (p_1, m)$ is a positive root* (*where m is the vertex nearest to p_1 among all vertices of $D(\alpha)$ that are greater than all vertices of the graph $D(\alpha^-)$ except p_1*).

This correspondence $f : \mathrm{Root}_1^* \to S(1,1,1)$ is given by the formula

$$f(\alpha) = \mathrm{rrWord}(\alpha - |q|) \cdot \mathrm{rrWord}(\beta).$$

§8. The set $S(1,2,1)$

Let v be a regular word with two occurrences of its leading letter p_1; we denote by v_1 and v_2 the regular subwords of v such that $v = v_1 \cdot v_2$ and p_1 occurs in each of the words v_1, v_2 exactly once. The subwords v_1, v_2 are determined uniquely.

Lemma 8.1. *A word u is an element of the set $S(1,2,1)$ if and only if $u = vw$, where*

1) *v, w are* RR-*words*;

2) *if $v_2 > w$, then the composition of the word $v_2 w$ is rooted.*

Proof. Let u be a word in $S(1,2,1)$.

1) By the definition of elements of the set $S(1,2,1)$, the word u is an SR-word of the form $u = vw$, where v and w are regular words with the same leading letter p_1 that occurs twice in v and only once in w; moreover, these words must satisfy the relation $w \boxdot v$.

Being regular proper subwords of the SR-word u, the words v and w are RR-words.

2) Since u is an SR-word, $r(u)$ is an RR-word by Remark 5.4. If $v_2 > w$, then the word $v_2 w$ is regular, because, by their choice, v_2 and w are regular words. Being a regular subword of the SR-word u, the word $v_2 w$ is an RR-word, and consequently, its composition is rooted.

Now we prove the converse statement. Let (v, w) be a pair satisfying conditions 1) and 2); we show that the word $u = vw$ is an element of $S(1,2,1)$.

The relation $w \boxdot v$ implies that $w < v$. Hence, by Remark 4.1, the word u is regular. Since the letter p_1 occurs three times in vw, the composition of vw is not rooted.

Since the regular words v and w have the same leading letter p_1, which occurs exactly once in w, we see that $l(u) = l(vw) = v$. Hence, $l(u)$ is an RR-word.

First, let $v_2 > w$. In this case the word $v_2 w$ is regular by Remark 4.1. By property 2), the composition of the word $v_2 w$ is rooted. Since v_2 and w are $(\mathrm{RR}, 1)$-words, they are regular and, by Remark 3.2, they begin with the only occurrences of the letter p_1 in them. Therefore, $l(v_2 w) = v_2$, $r(v_2 w) = w$. Hence, $l(v_2 w)$ and $r(v_2 w)$ are RR-words. Then, by Remark 5.3, we have $r(u) = v_2 w$.

If $v_2 \le w$, then $r(u) = w$. Thus, in both cases, $r(u)$ is an RR-word.

The facts proved above and Remark 5.4 show that the word $u = vw$ is an SR-word. Since property 1) is also fulfilled, we conclude that $u \in S(1,2,1)$. □

Let a and w be nonempty words; we shall write $w \boxed{<} a$ if $w \boxed{\prec} a$ or if a can be obtained from w by discarding the last letter.

We introduce the following notation: $P^*(2,1)$ is the set of all pairs of RR-words (v, w) with the same leading letter p_1 that occurs twice in v and once in w;

$P(2,1)$ is the set of all pairs in $P^*(2,1)$ that satisfy condition 1) of Lemma 8.1;

$P(1,1,<)$ is the set of all pairs of $(\mathrm{RR},1)$-words (a, w) with one and the the same leading letter and such that $w \boxed{<} a$.

Remark 8.1. A pair (v, w) belongs to $P(2,1)$ if and only if $(v_1, w) \in P(1,1,<)$.

We split the set $S(1,2,1)$ into the disjoint union of the following four subsets:

$S(1,2,1; z-, <)$ is the set of all words vw in $S(1,2,1)$ such that $(v, w) \in P^*(2,1)$, $|v| \in \mathrm{Root}(2, \max 1)$, and the word v_1 is obtained from w by discarding the last letter;

$S(1,2,1; z-, \prec)$ is the set of the words vw in $S(1,2,1)$ such that $(v, w) \in P^*(2,1)$, $|v| \in \mathrm{Root}(2, \max 1)$, and $w \boxed{\prec} v_1$;

$S(1,2,1; z+, <)$ is the set of all words vw in $S(1,2,1)$ such that $(v, w) \in P^*(2,1)$, $|v| \in \mathrm{Root}(2, \max 2)$, and the word v_1 is obtained from w by discarding the last letter;

$S(1,2,1; z+, \prec)$ is the set of all words vw in $S(1,2,1)$ such that $(v, w) \in P^*(2,1)$, $|v| \in \mathrm{Root}(2, \max 2)$, and $w \boxed{\prec} v_1$.

For a positive root α, we denote by $M^-(\alpha)$ the set of all simple roots β such that $\alpha + \beta$ is a root.

Lemma 8.2. *The elements of the set* $S(1,2,1; z-, <)$ *are in one-to-one correspondence with the roots in* $\mathrm{Root}(2, \max 1)$. *This correspondence* $f : \mathrm{Root}(2, \max 1) \to S(1,2,1; z+, <)$ *is given by the formula* $f(\alpha) = vv_1q$, *where* $v = \mathrm{rrWord}(\alpha)$, v_1 *is the* $(\mathrm{RR},1)$-*word whose composition is the sum of all double vertices of the graph* $D(\alpha)$, *and* $q = x_1^-(\alpha)$ *is the only single vertex of* $D(\alpha)$ *adjacent to a double vertex.*

Proof. Let u be a word in $S(1,2,1; z-, <)$, and let p_1 denote the leading letter of u. Let $u = vw$ be the decomposition as in Lemma 8.1, and let $v = v_1v_2$ be the decomposition of v such that v_1 and v_2 are regular subwords of v and the letter p_1 occurs in each of the words v_1, v_2 only once.

Then $|v_1| = [z, x_2^+(|v|)]$ and $|v_2| = [z, z_1(|v|)]$ by Lemma 4.7, and $w \boxed{<} v_1$ by Remark 8.1.

Since $u \in S(1,2,1; z-, <)$, the word v_1 can be obtained from w by discarding the last letter. Therefore, $|w| = |v_1| + |q|$, where q is the last letter of w. Hence, $q \in M^-(|v_1|)$. Since $|v_1| = [z, x_1^-(|v|))$, we have either $q = z$, or $q = x_1^-(|v|)$.

But $|v| \in \mathrm{Root}(2, \max 1)$, and from Lemma 3.3 it follows that no $(\mathrm{RR},1)$-word of composition $|v_1| + |q|$ can end with the letter z. Therefore, $q = x_1^-(|v|)$.

Let $\mu : S(1,2,1; z-, <) \to \mathrm{Root}(2, \max 1)$ be the mapping that sends any element u of the set $S(1,2,1; z-, <)$ to the root $|v|$.

We show that the mapping μ is injective. A root $|v|$ determines the letters $x_1^-(|v|)$ and $x_2^+(|v|)$ uniquely. Indeed, $x_1^-(|v|)$ is the only single vertex of $D(\alpha)$ adjacent to a double vertex, and $x_2^+(|v|)$ is the only double vertex of $D(\alpha)$ adjacent to a single vertex. Next, $|v_1| = [z, x_2^+(|v|)]$, $|w| = |v_1| + |q|$, and $q = x_1^-(|v|)$. Hence, the root $|v|$ determines the elements $|v_1|$ and $|w|$ uniquely; by Corollary 4.2, the RR-words v, v_1, w, and $u = vw$ are uniquely determined by $|v|$ together with $|v_1|$ and $|w|$. Thus, the mapping μ is injective.

Now, we show that the mapping μ is surjective. Let α be a root in $\mathrm{Root}(2, \max 1)$. We put $v = \mathrm{rrWord}(\alpha)$, $q = x_1^-(|v|)$, and $w = \mathrm{rrWord}([q, z])$. Let v_1 be the $(\mathrm{RR},1)$-word whose composition is equal to the sum of all double vertices of the graph $D(\alpha)$, and let v_2 be the $(\mathrm{RR},1)$-word whose composition is equal to the sum of all vertices of $D(\alpha)$. By Lemma 4.7, we have $v = v_1v_2$.

The word v_2 is regular; hence, v_2 begins with its leading letter p_1. Since $v = v_1 v_2$, we see that the word $v_1 p_1$ is a beginning of v.

Since $q = x_1^-(|v|)$ by Lemma 3.3 and $\alpha \in \mathrm{Root}(2, \max 1)$, we obtain $w = v_1 q$.

By the choice of α, the leading vertex p_1 of the graph $D(\alpha)$ is double. Therefore, the single vertex $q = x_1^-(\alpha)$ is smaller than p_1. Since the word $v_1 p_1$ is a beginning of v and $w = v_1 q$, we conclude that $w \mathrel{\boxed{\lessdot}} v$. Hence, the pair (v, w) is contained in the set $P(2, 1)$.

The relations $|w| = [z, x^-(|v|)]$ and $|v_2| = [z, z_1(|v|)]$ imply that $|w| \leq |v_2|$. Hence, $v_2 \leq w$ by Lemma 7.1. Therefore, condition 2) of Lemma 8.1 is fulfilled, and applying that lemma we see that the word vw is an element of the set $S(1, 2, 1)$. Thus, the mapping μ is surjective. We have already proved that μ is injective. Hence, the mapping μ is bijective.

It is not difficult to understand that μ and f are mutually inverse mappings. □

Lemma 8.3. *The set* $S(1, 2, 1; z-, \prec)$ *is empty.*

Proof. Assume the contrary. Let u be an element of $S(1, 2, 1; z-, \prec)$. By definition, we have $u = vw$, where $(v, w) \in P^*(2, 1)$, $|v| \in \mathrm{Root}(2, \max 1)$, and $w \mathrel{\boxed{\lessdot}} v_1$.

The word v is an $(\mathrm{RR}, 2)$-word, because $(v, w) \in P^*(2, 1)$. Since $|v| \in \mathrm{Root}(2, \max 1)$, Lemma 4.7 shows that $v = v_1 v_2$, where v_1, v_2 are $(\mathrm{RR}, 1)$-words, $|v_1| = [z, x_2^+(|v|)]$ is the sum of all double vertices of the graph $D(|v|)$, and $|v_2| = [z, z_1(|v|)]$ is the sum of all vertices of $D(|v|)$.

Since $(v, w) \in P^*(2, 1)$ and $w \mathrel{\boxed{\lessdot}} v_1$, the pair (v_1, w) is an element of $P(1, 1)$. Therefore, by Lemma 7.2, the element $|v_1| + |q|$ is a root (here q stands for the last letter of the word w), i.e., $q \in M^-(|v_1|)$. Hence, either the vertex q is the only single vertex $x_1^-(|v|)$ of $D(|v|)$ that has an adjacent double vertex, or $q = z$.

Let $q = x_1^-(|v|)$. Then $|v_1| + |q| = [q, z]$. Since $(v_1, w) \in P(1, 1)$, Lemma 7.5 shows that $p_1 \in (p_2(|v_1| + |q|), q)$, i.e., the vertex second by priority in the interval $[q, z] = [x_1^-(|v|), z]$ belongs to the interval $(p_1, z]$, and this contradicts the fact that $|v| \in \mathrm{Root}(2, \max 1)$. Thus, $q = z$.

Since q is the last letter of w, now Lemma 7.5 shows that the composition of the word $\widetilde{w}$ obtained from w by discarding the last letter coincides with $[z, m)$, where m is the vertex nearest to p_1 among all vertices of the interval $[x_1^-(|v|), p_1)$ that are greater than all vertices of the interval $(p_1, z]$.

Since $|\widetilde{w}| = [z, m)$ and $|v_2| = [z, z_1(|v|)]$, from Lemma 3.3 it follows that the word $\widetilde{w} \cdot m$ is a beginning of v_2. Moreover, $q < m$ by the choice of m, and $w = \widetilde{w} \cdot q$. Thus, $w \mathrel{\boxed{\lessdot}} v_2$, whence $w < v_2$. Since the words v_2, w are regular, so is the word $v_2 w$. Since $v_2 w$ is a regular proper subword of the SR-word u, the composition of $v_2 w$ is rooted.

On the other hand, we have $z \leq |v_2|$ because $|v_2| = [z, z_1(|v|)]$. Since $|w| = |z| + [z, m)$ and the interval $[z, m)$ is not empty (it contains the vertex p_1), we see that $2 \cdot z \leq |w|$. Hence, $3 \cdot z \leq |v_2| + |w|$.

Therefore, the element $|v_2| + |w|$ is not a root, a contradiction. □

We denote by $\mathrm{Root}'(2, \max 2)$ the set of all positive roots α in $\mathrm{Root}(2, \max 2)$ such that the interval $[m, p_1(\alpha)]$ has at least three elements. Here m is the vertex nearest to p_1 among all vertices of the interval $[z, p_1(\alpha))$ that are greater than all vertices of the interval $(p_1, x_1^-(\alpha)]$.

Lemma 8.4. *The elements of the set* $S(1, 2, 1; z+, <)$ *are in one-to-one correspondence with the positive roots* α *in* $\mathrm{Root}'(2, \max 2)$. *This correspondence* $f : \mathrm{Root}'(2, \max 2) \to S(1, 2, 1; z+, <)$ *is given by the formula* $f(\alpha) = \mathrm{rrWord}(\alpha) \cdot \mathrm{rrWord}([z, m'] + [z, x_2^+(|v|)])$, *where* m' *is the vertex of the interval* $[m, p_1]$ *adjacent to* m.

Proof. Let u be a word in $S(1,2,1;z+,<)$. By definition, we have $u = vw$, where $(v,w) \in P^*(2,1)$, $|v| \in \mathrm{Root}(2,\max 2)$, and the word v_1 is obtained from w by discarding the last letter.

Now, v is an $(\mathrm{RR},2)$-word, because $(v,w) \in P^*(2,1)$. Since $|v| \in \mathrm{Root}(2,\max 2)$, Lemma 4.7 shows that $v = v_1v_2$, where v_1, v_2 are $(\mathrm{RR},1)$-words, and $|v_1| = [z,m] + [z,x_2^+(|v|)]$, $|v_2| = (m, z_1(|v|)]$, where m is the vertex nearest to p_1 among all vertices of the interval $[z,p_1)$ that are greater than all vertices of the interval $(p_1, x_1^-(|v|)]$.

Since v_1 is obtained from w by discarding the last letter, we have $|w| = |v_1| + |q|$. Since the compositions of the words v_1, w are rooted, the element $|v_1|+|q|$ is a root, i.e., $q \in M^-(|v_1|)$. Since $|v_1| = [z,m]+[z,x_2^+(|v|)]$, the last letter q of w is equal either to m' or to $x_1^-(|v|)$.

First, we assume that $q = x_1^-(|v|)$. Then $|w| = [z,m] + [z,x_1^-(|v|)]$ because $|w| = |v_1| + |q|$.

Since the $(\mathrm{RR},1)$-word w ends with the letter q, we have $p_1 \notin [p_2(|w|), q]$ by Lemma 3.3.

Since $|v| \in \mathrm{Root}(2,\max 2)$, the relation $|w| = [z,m]+[z,x_1^-(|v|)]$ implies that $p_2(|w|) = p_2(|v_1|)$. But $p_1 \notin [p_2(|w|), q]$, hence $p_1 \notin [p_2(|v_1|), q]$. On the other hand, $p_1 \in [m, x_2^+(|v|)] \subseteq [p_2(|v_1|), q]$. This contradiction proves that $q \neq x_1^-(|v|)$. Therefore, $q = m'$.

At the same time $q < p_1$. Consequently, the interval $[m,p_1]$ contains at least three elements.

The identities $|w| = |v_1| + |q|$, $|v_1| = [z,m] + [z,x_2^+(|v|)]$, and $q = m'$ show that $|w| = [z,m'] + [z,x_2^+(|v|)]$.

It has been proved above that there is a mapping μ of the set $S(1,2,1;z+,<)$ to the set $\mathrm{Root}'(2,\max 2)$ such that $\mu(u) = |v|$. Since $|w| = [z,m'] + [z,x_2^+(|v|)]$, the root $|v|$ uniquely determines the root $|w|$. Therefore, by Corollary 4.2, the root $|w|$ uniquely determines the RR-words v and w, and consequently, the word $u = vw$. Thus, the mapping μ is injective.

We fix a root α in $\mathrm{Root}'(2,\max 2)$ and denote by v and w the only RR-words whose compositions are equal (respectively) to α and $[z,m'] + [z,x_2^+(|v|)]$. Then the word w ends with the letter m', and $v = v_1v_2$, where v_1, v_2 are $(\mathrm{RR},1)$-words such that $|v_1| = [z,m] + [z,x_2^+(|v|)]$ and $|v_2| = (m, z_1(|v|)]$.

Hence, the word v_1 is obtained from w by discarding the last letter. Therefore, $w \boxminus v$, i.e., requirement 1) of Lemma 8.1 is fulfilled.

The words v_1, v_2, and v_1v_2 are regular. By Remark 4.1, this implies that $v_2 < v_1$. Also, $|v_1| \nleq |v_2|$ because $|m| \leq |v_1|$ and $|m| \nleq |v_2|$. Hence, $v_2 \prec v_1$. Since $w = v_1q$, we obtain $v_2 \prec w$, whence $v_2 \leq w$. Thus, requirement 2) of Lemma 8.1 is also fulfilled. By that lemma, the word $u = vw$ is an element of $S(1,2,1)$.

Moreover, vw is an element of $S(1,2,1;z+,<)$, because $(v,w) \in P^*(2,1)$, $|v| \in \mathrm{Root}'(2,\max 2) \subseteq \mathrm{Root}(2,\max 2)$, and v_1 is obtained from w by discarding the last letter. This proves that the mapping μ is surjective.

Thus, the mapping μ is injective and surjective, i.e., it is bijective. It is easily seen that the mappings f and μ are mutually inverse. □

Lemma 8.5. *The elements of the set $S(1,2,1;z+,\prec)$ are in one-to-one correspondence with the positive roots α in* $\mathrm{Root}(2,\max 2)$. *The correspondence $f : \mathrm{Root}(2,\max 2) \to S(1,2,1;z+,\prec)$ is given by the formula $f(\alpha) = vw$, where v is the only* $(\mathrm{RR},2)$*-word whose composition is equal to α, w is the only* $(\mathrm{RR},1)$*-word whose composition is equal to $(m, x_1^-(\alpha)]$, and m is the vertex nearest to $p_1 = p_1(\alpha)$ among all vertices of the interval $(p_1, z]$ that are greater than all vertices of the interval $[x_2^+(\alpha), p_1)$.*

Proof. Let u be a word in $S(1,2,1;z+,\prec)$. By definition, we have $u \in S(1,2,1)$ and $u = vw$, where $(v,w) \in P^*(2,1)$, $|v| \in \mathrm{Root}(2,\max 2)$, and $w \boxminus v_1$. Since v is an

$(\mathrm{RR},2)$-word and $|v| \in \mathrm{Root}(2,\max 2)$, by Lemma 4.7 we obtain $v = v_1v_2$, where v_1 and v_2 are $(\mathrm{RR},1)$-words, $|v_1| = [z,m] + [z, x_2^+(|v|)]$, $|v_2| = (m, z_1(|v|)]$, and m is the vertex nearest to p_1 among all vertices of the interval $(p_1, z]$ that are greater than all vertices of the interval $[x_2^+(|v|), p_1)$.

Since v_1 and w are $(\mathrm{RR},1)$-words and $w \boxminus v_1$, the pair (v_1, w) is an element of $P(1,1)$. By Lemma 7.2, $|v_1| + |q|$ is a root, i.e., $q \in M^-(|v_1|)$. But $|v_1| = [z,m] + [z, x_2^+(|v|)]$, and consequently, q is equal either to $x_1^-(|v|)$, or to the only vertex m' of the interval $[m, p_1]$ that is adjacent to the vertex m.

First, assume that $q = m'$. Then $|v_1| + |q| = [z, m'] + [z, x_2^+(|v|)]$. Also, $\max[z, p_1) > \max(p_1, x_2^+(|v|)]$. Hence, the RR-word with composition $|v_1| + |q|$ ends with the letter q, which contradicts statement 3) of Lemma 7.3, because the pair (v_1, w) is an element of $P(1,1)$. This contradiction proves that $q \neq m'$.

Hence, $q = x_1^-(|v|)$. Then $|v_1| + |q| = [z,m] + [z, x_1^-(|v|)]$. Since the pair (v_1, w) is an element of $P(1,1)$, from Lemma 7.5 it follows that $|w| = (m, x_1^-(|v|)]$.

Let $\mu : S(1,2,1; z+, <) \to \mathrm{Root}(2, \max 2)$ be the mapping defined by the rule $\mu(u) = |v|$. Since $|w| = (m, x_1^-(|v|)]$, the root $|v|$ determines the root $|w|$. Hence, by Corollary 4.2, the root $|v|$ determines the RR-words v, w, and consequently, the word $u = vw$. Thus the mapping μ is injective.

Now we fix a root α $\mathrm{Root}(2, \max 2)$ and denote by v and w the RR-words the compositions of which are equal (respectively) to α and $(m, x_1^-(|v|)]$. Then the word w ends with the letter $q = x_1^-(|v|)$, and $v = v_1v_2$, where v_1 and v_2 are $(\mathrm{RR},1)$-words such that $|v_1| = [z,m] + [z, x_2^+(|v|)]$ and $|v_2| = (m, z_1(|v|)]$.

Let a be the $(\mathrm{RR},1)$-word the composition of which is equal to $(m, x_1^-(|v|))$. By Lemma 3.3, we have $w = aq$. The word am is a beginning of the word v_1, and $q < m$ by the choice of the element m. Hence, $w \boxminus v_1$.

Since the word v_1 is a beginning of v, it follows that $w \boxminus v$. Thus, the requirement 1) of Lemma 8.1 is fulfilled.

Since $|w| = (m, x_1^-(|v|)]$ and $|v_2| = (m, z_1(|v|)]$, we have $|w| \le |v_2|$. The words v_2 and w are RR-words, and now from Lemma 7.1 it follows that $w \ge v_2$. Thus, requirement 2) of Lemma 8.1 is also fulfilled. By that lemma, the word $u = vw$ is an element of the set $S(1,2,1)$.

But we know that $(v,w) \in P^*(2,1)$, $|v| \in \mathrm{Root}(2, \max 2)$, and $w \boxminus v_1$; therefore, the word $u = vw$ is an element of $S(1,2,1; z+, \prec)$ by the definition of this set. Thus, the mapping μ is surjective.

Since the mapping μ is injective and surjective, it is bijective. It is easily seen that the mappings f and μ are mutually inverse. □

Remark 8.2. The set $S(1,2,1)$ splits into the disjoint union of the subsets $S(1,2,1; z-, <)$, $S(1,2,1; z+, <)$, and $S(1,2,1; z+, \prec)$ that are described in Lemmas 8.2, 8.4, and 8.5.

§9. The sets $S(1,2,2)$, $S(2,1,1)$, $S(1,2,0)$, $S(2,1,0)$, $S(2,2,0)$, and $S(3,1,0)$

Lemma 9.1. *The set $S(1,2,2)$ is empty.*

Proof. Assume the contrary. Let u be a word in $S(1,2,2)$. Then u is an SR-word, and it can be represented as the product $u = vw$ of regular words v and w such that the graphs of these words have the same leading vertex p_1 and $w \boxminus v$.

Being regular proper subwords of the SR-word u, the words v and w are RR-words. Moreover, their leading letter occurs twice in each of these words. Hence, these words are $(\mathrm{RR},2)$-words.

Since the words v and w are regular, they begin with their leading letter p_1. Therefore, there are unique decompositions of these words in the form $v = v_1v_2$ and $w = w_1w_2$ such that each of the factors v_1, v_2, w_1, w_2 begins with the only occurrence of p_1. Thus, the words v_1, v_2, w_1, w_2 are regular. Since each of them is a subword of an RR-word, the words v_1, v_2, w_1, w_2 themselves are RR-words.

By Lemma 4.7, we have $|v_1| = [z, m_1) + [z, x_2^+(|v|)]$, $|v_2| = [m_1, z_1(|v|)]$, $|w_1| = [z, m_2) + [z, x_2^+(|w|)]$, and $|w_2| = [m_2, z_1(|w|)]$, where m_1, m_2 are vertices most distant from p_1 among all vertices of the interval $[z, p_1)$ such that $\max[m_1, p_1) < \max[x_1^-(|v|), p_1)$ and $\max[m_2, p_1) < \max[x_1^-(|w|), p_1)$. The intervals $[z, m_1)$, $[z, m_2)$ may be empty.

Since $w \mathrel{\boxed{\lhd}} v$, we see that $v_1 = w_1$ and $w_2 \mathrel{\boxed{\lhd}} v_2$. The identity $v_1 = w_1$ shows that the vertices m_1 and m_2 coincide. Put $m = m_1 = m_2$.

The words v_2, w_2 have the same leading letter, and $|v_2| = [m, z_1(|v|)]$, $|w_2| = [m, z_1(|w|)]$; now from Lemma 3.3 it follows that one of the $(\mathrm{RR}, 1)$-words v_2 and w_2 is a beginning of the other. But $w_2 \mathrel{\boxed{\lhd}} v_2$. This contradiction proves the lemma. □

We denote by $\mathrm{Root}(2, \max 2, d_1 1)$ the set of all positive roots $\alpha \in \mathrm{Root}(2, \max 2)$ such that the graph $D(\alpha)$ has only one single vertex.

Lemma 9.2. *The elements of the set $S(2,1,1)$ are in one-to-one correspondence with the positive roots α in $\mathrm{Root}(2, \max 2, d_1 1)$. This correspondence $f : \mathrm{Root}(2, \max 2, d_1 1) \to S(2,1,1)$ is given by the formula $f(\alpha) = \mathrm{rrWord}([z, m] + [z, x_2^+(\alpha)]) \cdot \mathrm{rrWord}(\alpha)$, where m is the vertex nearest to p_1 among the vertices of the interval $[z, p_1)$ that are greater than all vertices of the interval $(p_1, x_1^-(\alpha))$; $p_1 = p_1(\alpha)$.*

Proof. Let u be an element of $S(2,1,1)$. Then u is an SR-word of the form $u = v^2 w$, where v and w are regular words with the same leading letter p_1, which occurs exactly once in each of these words; moreover, the words v, w must satisfy the relation $w \mathrel{\boxed{\lhd}} v$.

Since the words v, w are regular and the leading letter p_1 occurs exactly once in each of them, these words begin with the only occurrence of the letter p_1.

The relation $w \mathrel{\boxed{\lhd}} v$ implies, in particular, that $w < v$. Hence, the word vw is regular. Being a regular proper subword of the SR-word $v^2 w$, the word vw itself is an RR-word. Since its leading letter p_1 appears in it exactly two times, the word vw is an $(\mathrm{RR}, 2)$-word.

Let α denote the composition of the word vw. Observe that $w \prec v$ because $w \mathrel{\boxed{\lhd}} v$. Applying statement 3) of Lemma 4.5 to vw, we see that $\alpha \in \mathrm{Root}(2, \max 2)$. Since $w \mathrel{\boxed{\lhd}} v$, statement 2) of the same lemma applied to the word vw shows that $x_1^-(\alpha)$ is the only single vertex of the graph $D(\alpha)$.

It has been proved above that the formula $\mu(u) = |vw|$ defines a mapping μ from the set $S(2,1,1)$ to the set $\mathrm{Root}(2, \max 2, d_1 1)$. By Corollary 4.2, the $(\mathrm{RR}, 2)$-word vw is uniquely determined by its composition $|vw|$. Next, by Lemma 4.7, the $(\mathrm{RR}, 1)$-word v, and, with it, the word $u = v^2 w$, are also determined by $|vw|$ uniquely. Hence, the mapping μ is injective.

To show that μ is a surjective mapping, we fix a root α in $\mathrm{Root}(2, \max 2, d_1 1)$ and denote by v and w the only $(\mathrm{RR}, 1)$-words whose compositions are equal (respectively) to $[z, m] + [z, x_2^+(\alpha)]$ and to $(m, x_1^-(\alpha)]$. Put $u = v^2 w$.

Since the root α is contained in the set $\mathrm{Root}(2, \max 2, d_1 1)$, we have $|v| + |w| = \alpha$. By Lemma 4.7, the word vw is the only $(\mathrm{RR}, 2)$-word whose composition is equal to α.

Each of the words v, w begins with the only occurrence of its leading letter p_1. Hence, $l(u) = l(v^2 w) = v$, $r(u) = r(v^2 w) = vw$. So, the words $l(u)$ and $r(u)$ are RR-words.

The letter p_1 occurs in u three times. Therefore, the composition of the word u is not rooted.

The word $u = v^2 w$ is regular, because $vw < v$. Above, it has been shown that the composition of u is not rooted and that $l(u)$ and $r(u)$ are RR-words. Now from Remark 5.4 it follows that u is an SR-word.

We denote by a the only $(\mathrm{RR}, 1)$-word with composition equal to $(m, x_2^+(\alpha)]$. Lemma

4.5 shows that $w = a \cdot x_1^-(\alpha)$ and that the word am is a beginning of v. Moreover, $m > x_1^-(\alpha)$ by the choice of the vertex m.

Hence, $w \boxdot v$. Recall that u is an SR-word, $u = v^2 w$, and the words v, w are regular. Therefore, the word u is an element of the set $S(2,1,1)$ by the definition of that set. But $\alpha = \mu(u)$. Thus, the mapping μ is surjective.

Since the mapping μ is injective and surjective, it is bijective. Keeping in mind Lemma 4.7, we can easily deduce that the mappings f and μ are mutually inverse. □

Lemma 9.3. *The words in* $S(1,2,0)$ *are in one-to-one correspondence with the positive roots* α *in* $\mathrm{Root}(2, \max 2)$ *such that* m *is greater than all vertices of the interval* $(p_1, z_1(\alpha)]$. *Here* m *stands for the vertex nearest to* p_1 *among all vertices of the interval* $[z, p_1)$ *that are greater than all vertices of the interval* $(p_1, x_1^-(\alpha)]$, *where* $p_1 = p_1(\alpha)$. *Such a correspondence* f *is given by the formula* $f(\alpha) = \mathrm{rrWord}(\alpha) \cdot m$.

Proof. Let u be a word in $S(1,2,0)$. Then u is an SR-word representable in the form $u = vq$, where v is an $(\mathrm{RR}, 2)$-word and q is a letter that is smaller than the leading letter p_1 of v.

By Lemma 4.7, the word v is the product $v = v_1 v_2$ of unique $(\mathrm{RR}, 1)$-words v_1, v_2 whose compositions are equal (respectively) to $[z, m') + [z, x_2^+(|v|)]$ and $[m', z_1(|v|)]$; here m' is the vertex most distant from p_1 among all vertices of the interval $[z, p_1)$ such that $\max[m', p_1) < \max(p_1, x_2^-(|v|)]$. The interval $[z, m')$ is empty if and only if $m' = z$.

The word v_2 begins with the only occurrence of its leading letter p_1. Since $q < p_1$, the word $v_2 q$ is regular. Being a regular proper subword of the SR-word u, the word $v_2 q$ is an RR-word. Next, the leading letter p_1 of $v_2 q$ occurs in this word only once. Therefore, $v_2 q$ is an $(\mathrm{RR}, 1)$-word.

It follows that the element $|v_2| + |q|$ is a root. Hence, $q \in M^-(|v_2|)$. Since $|v_2| = [m', z_1(|v|)]$, there are only two possibilities: 1) $m' \neq z$ and $q = m$; 2) not all of the vertices of the graph B_n are vertices of the graph $D(|v|)$, and q is the only vertex of the graph B_n that does not belong to $D(|v|)$ but is adjacent to a vertex of that graph.

In the second case, the composition of the word $u = vq$ is rooted; but this is impossible, because u is an SR-word. Hence, the first of the above cases occurs. In particular, this means that $|v| \in \mathrm{Root}(2, \max 2)$.

The $(\mathrm{RR}, 1)$-word $v_2 m$ ends with the letter m; therefore, by Lemma 3.3, m is greater than any of the vertices of the interval $(p_1, z_1(|v|)]$.

Earlier, it was proved that the mapping μ that takes any word u in $S(1,2,0)$ to the root $|v|$ is a mapping from $S(1,2,0)$ to the set of roots α in $\mathrm{Root}(2, \max 2)$ such that m is greater than any of the vertices of the interval $(p_1, z_1(\alpha)]$.

By Corollary 4.2, the root $|v|$ uniquely determines the RR-word v. The last letter q of u coincides with m. Hence, the root $|v|$ uniquely determines the word $u = vm$. Therefore, the mapping μ is injective.

To show that μ is surjective, we fix a root α in $\mathrm{Root}(2, \max 2)$, denote by v the only RR-word whose composition is equal to α, and put $u = vm$.

First, we check that u is an SR-word. Since $m < p_1$ and the word v begins with the letter p_1, we see that $m < v$. Therefore, the word $u = vm$ is regular.

The vertex m is double in the graph $D(|v|)$. Therefore, m occurs three times in the word $u = v_2 q = v_2 m$, and consequently, the composition of the word u is not rooted.

Since $u = vm$, we have $l(u) = v$, so that $l(u)$ is an RR-word.

The word v begins with its leading letter p_1, which occurs twice in this word; since $u = vm$ and $m < p_1$, we obtain $r(u) = v_2 m$.

From Lemma 4.7 it follows that $|v_2| = (m, z_1(\alpha)]$. Hence, $|v_2 m| = |v_2| + |m| = [m, z_1(\alpha)]$. We have $p_1 \in [m, z_1(\alpha)]$, and, by our assumption, m is greater than any of the vertices of the interval $(p_1, z_1(\alpha)]$. Therefore, by Lemma 4.7, the only RR-word a whose composition is equal to $|v_2 m|$ ends with the letter m. But v_2 is an $(\mathrm{RR}, 1)$-word, whence $a = v_2 m$ (see Corollary 4.2). Hence, the word $r(u) = v_2 m$ is an RR-word.

In accordance with Remark 5.4, from the facts proved above it follows that u is an SR-word. Moreover, $u = vm$, where v is a regular word with two occurrences of its leading letter p_1, and the letter m is smaller than p_1. Therefore, $u \in S(1, 2, 0)$, and this means that the mapping μ is surjective.

We have already proved that this mapping is injective. Thus, the mapping μ is bijective. Moreover, the mappings f and μ are mutually inverse. □

Lemma 9.4. *The elements of the set $S(2, 1, 0)$ are in one-to-one correspondence with the nonsimple positive roots α that satisfy the following condition*:

1) *if the vertex z is single, then the vertex p_2 of $D(\alpha)$ second by priority is contained in the interval $[z, p_1)$. Here $p_1 = p_1(\alpha)$.*

Such a correspondence f is given by the formula $f(\alpha) = \widetilde{a} \cdot a$, where a is the only RR-*word whose composition is equal to α, and $\widetilde{a}$ is the word obtained from a by discarding the last letter.*

Proof. Let u be a word in $S(2, 1, 0)$. Then it is an SR-word of the form $u = v^2 q$, where v is a regular word such that the leading vertex p_1 of its graph is single, and q is a letter such that $q < p_1$.

The word v is regular and the letter p_1 occurs in it only once; therefore, v begins with p_1.

Since $q < p_1$, the word vq is regular. Being proper regular subwords of the SR-word u, the words v and vq are RR-words. This means, in particular, that the compositions of the words v and vq are rooted.

The root $\alpha = |v| + |q|$ is not simple, because the word v is not empty.

Since u is an SR-word, the composition of u is not rooted. Hence, the root α satisfies condition 1).

We have shown above that the mapping μ that takes any word u in $S(2, 1, 0)$ to the element $|v| + |q|$ is a mapping from the set $S(2, 1, 0)$ to the set of nonsimple positive roots satisfying condition 1).

Any positive root α uniquely determines the RR-word a of composition α. In its turn, the word a uniquely determines its last letter q and the word v obtained from a by discarding the last letter; consequently, it uniquely determines the word $u = v^2 q$. Thus, the mapping μ is injective.

To show that the mapping μ is surjective, we fix a nonsimple root α, denote by a the only RR-word of composition α, and put $u = \widetilde{a} \cdot a$.

First we check that u is an SR-word. Since the root α satisfies condition 1), the composition of the word u is not rooted.

Any proper beginning of a word is greater than the word itself. Hence, $\widetilde{a} > a$. But the words a and $\widetilde{a}$ are regular, because their leading letters occur in each of them only once. Therefore, the word $u = \widetilde{a} \cdot a$ is regular.

The words a and $\widetilde{a}$ begin with the only occurrences of their leading letters, and $u = \widetilde{a} \cdot a$. Hence, $l(u) = \widetilde{a}$ and $r(u) = a$. But the words $\widetilde{a}$ and a are RR-words, and consequently, $l(u)$, $r(u)$ are RR-words.

In accordance with Remark 5.4, the facts proved in the preceding paragraphs imply that u is an SR-word. Recall that $u = v^2 q$, where $v = \widetilde{a}$ and q is the last letter of the word a. The word v is regular and $q < p_1$, because a regular word a that is not a letter cannot begin with its leading letter. Thus, the word u is an element of $S(2, 1, 0)$. This proves that the mapping μ is surjective.

Above, it was proved that the mapping μ is injective. Hence, it is a bijection. The mappings f and μ are mutually inverse. □

Lemma 9.5. *The elements of the set $S(2,2,0)$ are in one-to-one correspondence with the positive roots α such that the leading vertex p_1 of the graph $D(\alpha)$ is double and in this graph there are at least two single vertices. Such a correspondence f is given by the formula $f(\alpha) = \widetilde{a} \cdot a$, where a is the only* RR*-word of composition α and $\widetilde{a}$ is the word obtained from a by discarding the last letter.*

Proof. Let u be a word in $S(2,2,0)$. Then $u = v^2q$, where v is a regular word with exactly two occurrences of its leading letter p_1, and q is a letter that is smaller than p_1.

Since the word v is regular, the words v and v_2 begin with their common leading letter p_1. But $q < p_1$. Hence, $q < v$ and $q < v_2$. Therefore, the words vq and v_2q are regular. Being regular proper subwords of the SR-word u, the words v, vq, and v_2q are RR-words.

This means, in particular, that their compositions are rooted. We denote by α the root $|v| + |q|$.

By Lemma 4.7, the only $(\mathrm{RR}, 2)$-word whose composition is equal to α ends with the letter $z_1(\alpha)$. This letter is a single vertex of the graph $D(\alpha)$. Hence, the last letter q of the word u coincides with $z_1(\alpha)$, i.e., q is not a vertex of the graph $D(|v|)$, but q has two adjacent vertices in the latter graph. Therefore, the graph $D(\alpha)$ has at least two single vertices.

Earlier, it was proved that the mapping μ that takes any word u in $S(2,2,0)$ to the root $\alpha = |v| + |q|$ is a mapping from the set $S(2,2,0)$ to the set of all positive roots α such that the leading vertex of the graph $D(\alpha)$ is double and at least two vertices of this graph are single.

The root α uniquely determines the vertex $q = z_1(\alpha)$, and the composition $|v| = \alpha - |q|$ of the word v; next, by Corollary 4.2, it uniquely determines the word v itself, and consequently, the word $u = v^2q$. Thus, the mapping μ is injective.

To show that μ is surjective, we fix a root α satisfying the conditions of the lemma. We denote by a the only RR-word whose composition is equal to α, and by v the word obtained from a by discarding the last letter, and put $u = va$.

First, we check that u is an SR-word. The word a has two occurrences of its leading letter p_1. By Lemma 4.7, the last letter q of this word coincides with $z_1(\alpha)$. In particular, q is a single vertex of the graph $D(\alpha)$. Therefore, $q \neq p_1$. Hence, the leading letter p_1 of the word v occurs in v twice.

It follows that the letter p_1 occurs in the word $u = va$ four times, and consequently, the composition of the word u is not rooted.

Let $a = a_1a_2$ and $v = v_1v_2$ be decompositions with all factors a_1, a_2, v_1, and v_2 beginning with the only occurrences of the letter p_1 in them. It is clear that $a_1 = v_1$ and that the word v_2 is obtained from the word a_2 by discarding the last letter.

We show that $l(u) = v$. Indeed, $u = v^2q$ and the word v is regular. Hence, v is a beginning of the word $l(u)$. Suppose that $l(u) \neq v$. Then v is a proper beginning of the word $l(u)$, i.e., $l(u) = vb$, where b is a nonempty word. But $u = v^2q$ and $l(u)$ is a proper beginning of u. Hence, b is a beginning of v, so that $v \leq b$. But the word $l(u) = vb$ is regular, and this contradicts Remark 4.1. Thus, $l(u) = v$, and $l(u)$ is an RR-word.

We prove that $r(u) = a$. Indeed, otherwise $r(u) = v_2v_1v_2q$, because the regular word $r(u)$ must begin with its leading letter. Since the word $r(u)$ is regular, its cyclic rearrangement $v_1v_2qv_2$ is smaller than $v_2v_1v_2q$. But the words v_1 and v_2 begin with the only occurrences of their common leading letter p_1. It follows that $v_2 \leq v_1$. But $v_2 < v_1$ because v_1v_2 is a regular word. This contradiction proves that $r(u) = a$, and consequently, $r(u)$ is an RR-word.

In accordance with Remark 5.4, the facts proved in the preceding paragraphs imply that u is an SR-word. Since $u = v^2 q$, the word v is regular, and q is a letter, we see that u is an element of the set $S(2, 2, 0)$.

Thus, the mapping μ is surjective. We have already proved that this mapping is injective. Therefore, μ is bijective, and it is clear that μ is inverse to f. □

Lemma 9.6. *The elements of the set $S(3, 1, 0)$ are in one-to-one correspondence with the vertices q of the graph B_n such that*

1) *q is distinct from z and from the leading vertex p_1 of the interval $[z, q]$;*

2) *the vertex of the interval $[z, q]$ second by priority belongs to the interval $(p_1, q]$.*

Such a correspondence is given by the formula $f(q) = v^3 \cdot q$, where v is the only RR-*word whose composition is equal to $[z, q)$.*

Proof. Let u be an element in $S(3, 1, 0)$. Then u is an SR-word of the form $u = v^3 q$, where v is a regular RR-word with one appearance of its leading letter p_1.

The regular word v begins with its leading letter p_1. The word u cannot end with its leading letter p_1, because it is regular and its length is at least two. Hence, $q < p_1$. Therefore, the word vq is regular. Since $vq < v$, the word $v^2 q$ is regular. Being a regular proper subword of the SR-word u, the word $v^2 q$ is an RR-word.

In particular, this implies that the composition of the word $v^2 q$ is rooted. We denote $|v^2 q|$ by α. The graph of the root α has only one single vertex. Therefore, $|v| = [z, q)$ by Lemma 4.7; thus, the vertex q satisfies conditions 1) and 2).

Above, it was proved that there exists a mapping μ from the set $S(3, 1, 0)$ to the set of the vertices of the graph B_n that satisfy conditions 1) and 2). This mapping takes an element u in $S(3, 1, 0)$ to its last letter q.

Since $|v| = [z, q)$, the letter q uniquely determines the composition of the word v, and by Corollary 4.2, it determines the word v itself, and consequently, the word $u = v^2 q$. Thus, the mapping μ is injective.

To show that μ is surjective, we fix a vertex q of the graph B_n for which conditions 1) and 2) are fulfilled. Let v denote the RR-word whose composition is equal to $[z, q)$, and let u be the word $v^3 q$.

First, we check that u is an SR-word. The leading letter p_1 of v occurs in this word. Hence, the letter p_1 occurs in the word u three times. Therefore, the composition of u is not rooted.

By condition 1), we have $q < p_1$. Hence, the word $u = v^3 q$ is regular.

We show that $l(u) = v$. Indeed, the word v is regular; therefore, by definition, v is a beginning of the word $l(u)$. Assume that $l(u) \neq v$. Then $l(u) = vb$, where b is a nonempty word. But $l(u)$ is a proper beginning of u. Hence, b is a beginning of v^2. The word vb is regular, but the words v^2 and v^3 are not regular. Therefore, $b = b'$ or $b = vb'$, where b' is a proper beginning of v. The word v begins with the only occurrence of its leading letter p_1. Hence, $b'v > vb'$ and $vb'v > vb'v$; consequently, the cyclic rearrangement bv of the word vb is greater than vb. But, by assumption, the word vb is regular. We arrive at a contradiction. Thus, we have $l(u) = v$, so that $l(u)$ is an RR-word.

The word v begins with the only occurrence of its leading letter p_1. Hence, the leading letter p_1 of the word $u = v^3 q$ occurs in this word more than once. So, the word $r(u)$ begins with the letter p_1. Since the word $v^2 q$ is regular, we have $r(u) = v^2 q$. Now Lemma 4.7 and conditions 1) and 2) imply that $r(u)$ coincides with the only RR-word whose composition is equal to $[z, q] + [z, q)$. Thus, $r(u)$ is an RR-word.

Remark 5.4 and the facts proved above show that u is an SR-word. But $u = v^3 q$, the word v is regular, and q is a letter that is smaller than p_1. Therefore, u is an element of the set $S(3, 1, 0)$. So, μ is a surjective mapping.

We have already seen that the mapping μ is injective. Hence, it is bijective. It is easily seen that μ is inverse to f. □

Theorem 9.1. *The set of all* SR*-words splits into the disjoint union of the following* 14 *subsets*: $S(len{=}2)$, $S(q{<}p_2)$, $S(q{=}p_2)$, $S(p_21)$, $S(p_22)$, $S(1,1,1)$, $S(1,2,1;z-,<)$, $S(1,2,1;z+,<)$, $S(1,2,1;z+,\prec\ )$, $S(2,1,1)$, $S(1,2,0)$, $S(2,1,0)$, $S(2,2,0)$, $S(3,1,0)$, *which were described* (*respectively*) *in Remarks* 6.2, 6.3, 6.4, *Lemmas* 6.1, 6.2, *Corollary* 7.2, *and Lemmas* 8.2, 8.4, 8.5, 9.2–9.6.

Theorem 9.2. *Let X be the set of generators of the Lie algebra B_n^+ that correspond to simple roots. For any ordering of the set X, the minimal Gröbner–Shirshov basis of this algebra coincides with the set of relations of the form $[u]=0$, where u runs through the set of the words described in Theorem* 9.1.

Theorem 9.2 follows immediately from Corollary 4.2, statement 3) of Lemma 1.2, and Theorem 9.1. Essentially the same statements imply the following interesting generalization of Theorem 9.1.

Theorem 9.3. *Let $L=\bigoplus_{\pi\in ZX} L_\pi$ be a Lie algebra with generators in an n-element set X such that the ideal of relations among these generators is homogeneous with respect to each generator. Assume that, for any element α of the group ZX, the homogeneous component L_α does not reduce to zero if and only if the corresponding component of the algebra B_n^+ does not reduce to zero. Then L is isomorphic to the algebra B_n^+* (*moreover, there exists an isomorphism that does not move the elements of the set X*). *In particular, the algebras L and B_n^+ have the same MGShB* (*for any ordering of the generators from X*).

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