International Journal of Algebra, Vol. 5, 2011, no. 12, 579 - 589

The Symmetric Algebra for Certain Monomial Curves[1]

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Abstract. Let $p \geq 2$ and $0 < m_0 < m_1 < \ldots < m_p$ be a sequence of positive integers such that they form a minimal arithmetic sequence. Let $\wp$ denote the defining ideal of the monomial curve $\mathcal{C}$ in $\mathbb{A}_K^{p+1}$, defined by the parametrization $X_i = T^{m_i}$ for $i \in [0, p]$. Let R denote the polynomial ring $K[X_1, \ldots, X_p, X_0]$. In this article, we construct a minimal Gröbner basis for the symmetric algebra for such curves, as an R-module and what is interesting is that the proof does not require any S-polynomial computation.

Mathematics Subject Classification: 13P10, 13A30

Keywords: Monomial Curves, Gröbner Basis, Symmetric Algebra

1. Notation

Let $\mathbb{N}$ denote the set of non-negative integers and the symbols $a, b, d, i, i', j, j', l, l', m, n, p, q, s$ denote non-negative integers. For our convenience we define $[a\,,\,b] = \{i \mid a \leq i \leq b\}$,

$$\epsilon(i\,,\,j) = \begin{cases} i+j & \text{if} \quad i+j < p \\ p & \text{if} \quad i+j \geq p \end{cases} \quad \text{and} \quad \tau(i\,,\,j) = \begin{cases} 0 & \text{if} \quad i+j < p \\ p & \text{if} \quad i+j \geq p \end{cases}$$

2. Introduction

A class of rings, collectively designated *Blowup Algebras*, appear in many con-

[1]Research supported by U.G.C Minor Research Project No.F. PSW-033/10-11 (ERO)

structions in Commutative Algebra and Algebraic Geometry. The ancestor of the blowup algebra is the *symmetric algebra*. Given an R-module M, the *symmetric algebra* of M is an R-algebra $S(M)$ which together with an R-module homomorphism

$$\pi : M \longrightarrow S(M)$$

solves the following universal problem: For a commutative R-algebra B and any R-module homomorphism $\varphi : M \longrightarrow B$, there exists a unique R-algebra homomorphism $\overline{\varphi} : S(M) \to B$ such that $\varphi = \overline{\varphi} \circ \pi$. Thus, if M is a free module then $S(M)$ is the polynomial ring $R[T_1, \ldots, T_m]$,
where $m = \text{rank}(M)$. More generally, when M is given by the presentation

$$R^r \xrightarrow{\varphi} R^m \longrightarrow M \longrightarrow 0, \quad \varphi = (a_{ij}),$$

its symmetric algebra is the quotient of the polynomial ring $R[T_1, \ldots, T_m]$ by the ideal generated by the 1-forms

$$f_j = a_{1j}T_1 + \cdots + a_{mj}T_m, \quad j = 1, \ldots, r.$$

Conversely, any quotient ring of the polynomial ring $R[T_1, \ldots, T_m]/\mathcal{L}$, with $\mathcal{L}$ generated by the 1-forms in the T_i's is the symmetric algebra of a module.

3. Monomial Curves

Let $p \geq 2$ and $0 < m_0 < m_1 < \ldots < m_p$ is an arithmetic sequence of integers with $m_i = m_0 + id$ for $i \in [1, p]$ and $d \geq 1$. We also assume that $m_0 = ap + b$ with $a \geq 1$ and $b \in [1, p]$. Let Γ denote the numerical semigroup generated by $m_0, \ldots, m_p$ i.e., $\Gamma := \sum_{i=0}^{p} \mathbb{N} m_i$. We further assume that $\gcd(m_0, d) = 1$ and the set $S = \{m_0, \ldots, m_p\}$ forms a minimal set of generators for Γ.
Let K be a field and $X_1, \ldots, X_p, X_0, T$ are indeterminates. Let $\wp$ denote the kernel of the K-algebra homomorphism $\eta : R := K[X_1, \ldots, X_p, X_0] \to K[T]$, defined by $\eta(X_i) = T^{m_i}$ for $i \in [0, p]$. The prime ideal $\wp$ is an one-dimensional perfect ideal and it is the defining ideal of the affine monomial curve given by the parametrization $X_i = T^{m_i}$ for $i \in [0, p]$. It is easy to verify that $\wp$ is generated by binomials of the form $X_1^{\alpha_1} \cdots X_p^{\alpha_p} X_0^{\alpha_0} - X_1^{\beta_1} \cdots X_p^{\beta_p} X_0^{\beta_0}$ with $\displaystyle\sum_{i=0}^{p} \alpha_i m_i = \sum_{i=0}^{p} \beta_i m_i$.
The structure of the semigroup Γ was given by Patil & Singh (1990) under a more general assumption of almost arithmetic sequence on the integers $m_0 < m_1 < \ldots < m_p$. Subsequently, Patil (1993) constructed a minimal generating set $\mathcal{G}$ for $\wp$ which was proved to be a Gröbner basis by Sengupta (2003a). Al-Ayyoub(2009) pointed out a gap in Sengupta's proof(2003a) in one particular case. However, it is still not known whether the minimal generating set $\mathcal{G}$

for $\wp$ in that particular case is a Gröbner basis with respect to some suitable monomial order.

We restrict our attention only in the case of an arithmetic sequence. A complete description of all the syzygy modules was given by Sengupta (2003b), when $p = 3$ and $m_0 < m_1 < \ldots < m_p$ forms an arithmetic sequence. An explicit description of a minimal generating set $\mathcal{G}$ for $\wp$ case of arithmetic sequence is given in Sengupta (2000) and in Maloo-Sengupta (2003). We recall the set putting $Y = X_p$:

$$\mathcal{G} = \{\xi_{i,j} \mid i,j \in [1,p-2]\} \cup \{\phi_i \mid i \in [0,p-2]\} \cup \{\psi_{b,j} \mid j \in [0,p-b-1]\} \cup \{\theta\}$$

where

- $\xi_{i,j} = \begin{cases} X_iX_j - X_{i+j}X_0 & \text{if} \quad i+j \leq p-1 \\ X_iX_j - X_{i+j+1-p}X_{p-1} & \text{if} \quad i+j \geq p \end{cases}$
- $\phi_i = X_{i+1}X_{p-1} - X_iX_p$, $\psi_{b,j} = X_{b+j}X_p^a - X_jX_0^{a+d}$ and $\theta = X_p^{a+1} - X_{p-b}X_0^{a+d}$

Let us construct the set $\mathcal{G}' = \{\phi(i\,,\,j) \mid i,j \in [1,p-1]\} \cup \{\psi(b\,,\,i) \mid i \in [0,p-b]\}$ where

$$\phi(i\,,\,j) = X_iX_j - X_{\epsilon(i\,,\,j)}X_{i+j-\epsilon(i\,,\,j)} \quad \text{and} \quad \psi(b\,,\,i) = X_{b+i}X_p^a - X_iX_0^{a+d}$$

It is easy to verify that

- $\xi_{i,j} = \phi(i\,,\,j)$ if $i,j \in [1,p-2]$ and $i+j \leq p-1$
- $\xi_{i,j} + \phi_{i+j-p} = \phi(i,j)$ if $i,j \in [1,p-2]$ and $i+j \geq p$
- $\phi_i = \phi(i+1,p-1)$ if $i \in [0,p-2]$
- $\psi_{b,i} = \psi(b\,,\,i)$ if $i \in [0,p-b-1]$ and $\theta = \psi(b\,,\,p-b)$

Therefore, the set $\mathcal{G}$ is contained in the ideal generated by the set $\mathcal{G}'$ as well as their cardinalities are equal. Also note that $\mathcal{G}' \subseteq \wp$. Hence, $\mathcal{G}'$ is a minimal generating set for the ideal $\wp$.

Our aim is to describe a minimal Gröbner basis for the symmetric algebra for $\wp$. We start by proving that $\mathcal{G}'$ is a Gröbner basis with respect to a suitable monomial order in Section 4. This is necessary for computing the generators of the symmetric algebra in the subsequent sections, which is the main theme of this article. We construct a set of linear relations among the binomials in $\mathcal{G}'$. Finally we prove that the set is a minimal Gröbner basis for the first syzygy module of $\wp$ with respect to a suitable monomial order in Section 5. We use numerical method to show that the leading monomials of the generating sets indeed generate the initial ideals of the defining ideals. The advantage of this method is that it does not require any S-polynomial computation. In this regard we refer to Conca, A., Herzog, J., Valla, G. (1996). We use the notations LT and LM to mean leading term and leading monomial. For details of Gröbner bases we refer to Eisenbud (1995) and Cox-Little-O'Shea (1996). This work initiates the process of computing all relations (linear as well as non-linear) among the binomials in $\mathcal{G}$, culminating in a structure of the Rees algebra of $\wp$ (Mukhopadhyay & Sengupta (2009)), which is essential for understanding smoothness of the symbolic blow-ups of such curves (Mukhopadhyay & Sengupta (2009)).

4. Gröbner basis for $\wp$

Every monomial of R can be expressed in the form $\left(\prod_{i=1}^{p} X_i^{\alpha_i}\right) X_0^{\alpha_0}$. We identify the monomial, $\left(\prod_{i=1}^{p} X_i^{\alpha_i}\right) X_0^{\alpha_0}$ with the ordered tuple $(\alpha_1, \ldots, \alpha_p, \alpha_0) \in \mathbb{N}^{p+1}$. Let us define a weight function ω

on the monomials of R by the following :

- $\omega(X_i) = m_i$; $i \in [0, p]$.
- $\omega(fg) = \omega(f) + \omega(g)$; for any two monomials f and g of R.

We say that $f = \prod_{i=1}^{p} X_i^{\alpha_i} X_0^{\alpha_0} >_R g = \prod_{i=1}^{p} X_i^{\alpha'_i} X_0^{\alpha'_0}$ if and only if one of the following holds:

- $\omega(f) > \omega(g)$.
- $\omega(f) = \omega(g)$ and the right-most non-zero entry in the difference $(\alpha_1 - \alpha'_1, \ldots\ldots, \alpha_p - \alpha'_p, \alpha_0 - \alpha'_0)$ is negative .

Remark 4.1. *Let f and g are two monomials of R. One can easily check that:*

- $f - g \in \wp \iff \eta(f - g) = 0 \iff \omega(f) = \omega(g)$
- $\omega(X_i) \neq \omega(X_j)$ *for* $i \neq j$ *and* $i, j \in [0, p]$
- $\omega(X_i) + \omega(X_j) = \omega(X_0) + \omega(X_{i+j})$ *for* $i + j < p$ *and* $i, j \in [1, p-1]$
- $\omega(X_i) + \omega(X_j) = \omega(X_p) + \omega(X_{i+j-p})$ *for* $i + j \leq p$ *and* $i, j \in [1, p-1]$

Lemma 4.2. *Let m be the smallest integer which satisfies the relation $mm_p = nm_0 + m_i$ with $m, n \geq 1$ and $0 \leq i < p$ then : $m = a + 1, i = p - b$ and $n = a + d$.*

Proof. $mm_p = nm_0 + m_i \Longrightarrow mpd = (n-m+1)m_0 + id \Longrightarrow mp - i = 0(\text{modulo} \quad m_o)$ since $\gcd(m_0, d) = 1$. There exist $q \in \mathbb{N}$ such that $mp - i = qm_o$. Note that $q = 0$ implies $mp = i$ which is absurd and so $q \geq 1$. Therefore, $mp - i = q(ap+b)$ with $q \geq 1$ and hence $m \geq a+1$ since, $0 \leq i < p$ and $b \in [1, p]$. At this point note that $(a+1)m_p = (a+d)m_0 + m_{p-b}$. Hence, $m = a+1$, $n = a+d$ and $i = p-b$. □

Corollary 4.3. *Let n be the smallest integer which satisfies the condition $nm_0 = mm_p + m_i$ with $m, n \geq 1$ and $0 < i \leq p$ then : $n = a+d$, $i = b$ and $m = a$.*

Proof. Note that $mm_p + m_i = nm_0 \Longrightarrow (m+1)m_p = (n-1)m_0 + m_0 + (p-i)d \Longrightarrow (m+1)m_p = (n-1)m_0 + m_{p-i}$. Rest of the proof follows from Lemma 4.2. □

Lemma 4.4. *If we assume that $m \neq n \geq 0$ and $i \neq j \in [0, p]$ and $l \in \{0, p\}$ then $\omega(X_l^n X_i) \neq \omega(X_l^m X_j)$.*

Proof. $\omega(X_l^n X_i) = \omega(X_l^m X_j) \Longrightarrow$ either $sm_l + m_i = m_j$ or $sm_l + m_j = m_i$ for some s. This contradicts that Γ is minimally generated by the set S. Hence the proof.

Lemma 4.5. *If $g = X_0^m X_j$ with $m \geq 0$ and $j \in [1, p-1]$ and*

- $f_1 = X_p^n X_i$ *with* $i \in [1, p-1]$ and $1 \leq n \leq a-1$
- $f_2 = X_p^n X_i$ *with* $i \in [1, b-1]$ and $1 \leq n \leq a$

then $g - f_q \notin \wp$ for $q = 1, 2$.

Proof. $\omega(X_p^n X_i) = \omega(X_0^m X_j)$ implies that there exist $s \geq 0$ such that either $nm_p = mm_0 + sd$ or $nm_p + sd = mm_0$. Rest of the proof follows from Lemma 4.2 and Corollary 4.3. □

Lemma 4.6. *If $g = X_0^m X_p^n$ with $m \geq 0$ and $n \in [0, a]$ and*

- $f_1 = X_p^l X_i \quad 1 \leq l \leq a-1$ and $i \in [1, p-1]$
- $f_2 = X_0^l X_i \quad l \geq 0$ and $i \in [1, p-1]$
- $f_3 = X_p^l X_i \quad 1 \leq l \leq a$ and $i \in [1, b-1]$

then $g - f_q \notin \wp$ for $q = 1, 2, 3$.

Proof. The following cases will arise depending on m, n, l :

1. $m_i \in \langle S \setminus m_i \rangle$
2. $n(ap+b) = (m+1)(ap+b) + mpd + id$

3. $m(ap+b) = (n+1)(ap+b) + id$

Case(i) contradicts that Γ is minimally generated by the set S . Lemma 4.2 and Corollary 4.3 take care of Case(ii) and Case(iii). Hence the proof. $\square$

Theorem 4.7. *The set $\mathcal{G}'$ is a Gröbner Basis for $\wp$ with respect to $>_R$.*

Proof. If f is a monomial of R and $f \notin \mathrm{LT}(\mathcal{G}') = \{X_i X_j \mid i\ j \in [1\,,\,p-1]\} \cup \{X_{b+i} X_p^a \mid i \in [0, p-b]\}$ then f must be of the following form:

- $X_p^m X_i \; : \; 1 \le m \le a-1$ and $i \in [1, p-1]$
- $X_0^m X_i \; : \; i \in [1, p-1]$ and $m \ge 0$ is an integer
- $X_0^m X_p^n \; : \; 0 \le n \le a$ and $m \ge 0$ is an integer
- $X_p^m X_i \; : \; 1 \le m \le a$ and $i \in [1, b-1]$

Let $f\,,\,g \notin \mathrm{LT}(\mathcal{G}')$ are two distinct monomials of R. Now apply Lemma 4.4 to Lemma 4.6 to conclude that $f - g \notin \wp$. Therefore $\mathrm{LT}(\wp) = \langle\, \mathrm{LT}(\mathcal{G}')\,\rangle$. Hence the proof. $\square$

Theorem 4.8. *The set $\mathcal{G}'$ is a minimal Gröbner Basis for $\wp$ with respect to $>_R$.*

Proof It is enough to note that no two distinct elements of $\mathrm{LT}(\mathcal{G}')$ can divide each other. $\square$

5. Symmetric Algebra

Let $\widehat{R}$ denote the polynomial ring $K[\mathbb{X}\,,\,\Psi_b\,,\,\Phi]$ with old indeterminates $\mathbb{X} = \{X_1, \ldots, X_p, X_0\}$

and the new ones of the set $\Psi_b = \{\Psi(b\,,\,j) \mid j \in [0, p-b]\}$ and the set $\Phi = \cup_{i=1}^{p-1} \Phi(\text{p-i})$ such that

$\Phi(\text{i}) = \{\Phi(i\,,\,i), \Phi(i-1\,,\,i), \ldots, \Phi(1\,,\,i)\}$ for every $i = 1, \ldots, p-1$.

Let $R[t]$ be a polynomial ring with indeterminate t and $\varphi : \widehat{R} \longrightarrow R[t]$ is a K-algebra homomorphism defined by

- $\varphi(\Psi(b\,,\,i)) = \psi(b\,,\,i)t$ for all $i \in [0, p-b]$
- $\varphi(\Phi(i\,,\,j)) = \phi(i\,,\,j)t$ for all $i, j \in [1, p-1]$
- $\varphi(r) = r$ for all $r \in R$

Hence the Symmetric Algebra of the ideal $\wp$ is the polynomial ring $\widehat{R}/\mathcal{L}$ where the ideal $\mathcal{L} \subseteq \mathrm{Kernel}(\varphi)$ is generated by the set of polynomials which are linear with respect to the variables $\Psi(b\,,\,i)$ and $\Phi(i\,,\,j)$.

Let us write $\mathcal{M} = \{ \sum_{i,j} f_{i,j}\,\Phi(i\,,\,j) + \sum_{l} f_l \Psi(b\,,\,l) \quad : \quad f_{i,j}\ ,\ f_l \in R\}.$

It is easy to check that under usual addition and scalar multiplication $\mathcal{M} \subseteq \widehat{R}$ is a module over R. The module $\mathcal{N} = \mathcal{M} \cap \mathrm{Kernel}(\varphi)$ is a submodule of $\mathcal{M}$ and is called the first syzygy module of the ideal $\wp$. First agree that every monomial of $\mathcal{M}$ is of the form $\mathbb{X}^{\alpha}\mathbf{e}$ where either $\mathbf{e} = \Psi(b\,,\,i)$

or $\mathbf{e} = \Phi(i\,,\,j)$. Hence, every element H of $\mathcal{M}$ can be expressed uniquely as $H = \sum_i \mathbb{X}^{\alpha^i}\mathbf{e}_i$ where, $\alpha^i = (\alpha_{1(i)}\,,\ldots,\,\alpha_{p(i)}, \alpha_{0(i)}) \in \mathbb{N}^{p+1}$ and either

$\mathbf{e}_i = \Psi(b\,,\,l(i))$ or $\mathbf{e}_i = \Phi(l(i)\,,\,j(i))$ with

$l(i)\,,\,j(i) \in \mathbb{N}$. Note that this is a finite sum because every element H of $\mathcal{M}$ is a polynomial in $\widehat{R}$.

Before we proceed further, let us record the following remark.

Remark *It is interesting to note that* $\phi(i\,,\,j) = \phi(j\,,\,i)$ *for all* $i, j \in \mathbb{N}$. *Therefore,* $\varphi(\Phi(i\,,\,j)) =$

$\varphi(\Phi(j\,,\,i))$ *for all* $i, j \in [1, p-1]$. *Henceforth, throughout the rest of this section we write* $\phi(i,j), \Phi(i,j)$

to mean that $i \leq j$.

Let us define a function ϖ on the set of monomials of $\mathcal{M}$ by

$$\varpi(\mathbb{X}^{\alpha}\Psi(b\,,\,i)) = \mathbb{X}^{\alpha}X_p^{a}X_{b+i} \quad ; \quad \text{with} \quad i \in [0\,,\,p-b]$$

and $\varpi(\mathbb{X}^{\alpha}\Phi(i\,,\,j)) = \mathbb{X}^{\alpha}X_iX_j \quad ; \quad \text{with} \quad i\,,\,j \in [1\,,\,p-1]$.

We now define a monomial order $>_{\mathcal{M}}$ on $\mathcal{M}$ by $\mathbb{X}^{\alpha}\mathbf{e} >_{\mathcal{M}} \mathbb{X}^{\alpha'}\mathbf{e}'$ iff one of the following holds :

1. $\varpi(\mathbb{X}^{\alpha}\mathbf{e}) \quad >_R \quad \varpi(\mathbb{X}^{\alpha'}\mathbf{e}')$
2. $\varpi(\mathbb{X}^{\alpha}\mathbf{e}) = \varpi(\mathbb{X}^{\alpha'}\mathbf{e}')$, and one of the following holds :

   - $\mathbf{e} = \mathbb{X}^{\alpha}\Psi(b\,,\,i)$ and $\mathbf{e}' = \mathbb{X}^{\alpha'}\Psi(b\,,\,j)$ and $i < j$
   - $\mathbf{e} = \mathbb{X}^{\alpha}\Psi(b\,,\,i)$ and $\mathbf{e}' = \mathbb{X}^{\alpha}\Phi(l\,,\,j)$
   - $\mathbf{e} = \mathbb{X}^{\alpha}\Phi(i\,,\,j)$ and $\mathbf{e}' = \mathbb{X}^{\alpha}\Phi(i'\,,\,j')$ and either $j > j'$ or $j = j'$ and $i > i'$

Remark 5.1. *It is interesting to note that* $\varpi(F) = \mathrm{LM}(\varphi(F))$ *for every monomial* F *of* $\mathcal{M}$ *.*

Definition 5.2. An element $H \neq 0$ in $\mathcal{M}$ is called a *relation* if $\varphi(H) = 0$. A *relation* H is called a *reduced relation* if $H = FG$ with $F \in R$, $G \in \mathcal{M}$ and $\varphi(G) = 0$ implies that $F = 1$. A *reduced relation* H is called a *basic relation* if $H = F + G$ with $\varphi(F) = 0$ and $\varphi(G) = 0$ implies that either $F = 0$ or $G = 0$.

Remark 5.3. *Note that, by the definition of* φ *, every relation can be written as a finite linear combination of basic relations over* R*. Therefore, we may assume that the generators of the module* $\mathcal{N}$ *over* R *are basic relations. Henceforth, the term relation stands for a basic relation.*

For our convenience from now on we will use underlined terms to represent leading terms. Unless otherwise specified, the symbols H_l, c_l for each integer l will represent a monomial of $\mathcal{M}$ and an element of K respectively. Throughout the rest of this article, α_0 , $\alpha_p \in \mathbb{N}$ and α , $\beta \in \mathbb{N}^{p+1}$.

Lemma 5.4. *If* $H = \underline{c_1 H_1} + \sum_l c_l H_l$ *is a relation, then there exist* l' *such that* $\mathrm{LM}(\varphi(H_1)) = \mathrm{LM}(\varphi(H_{l'}))$ *with* $H_{l'} \neq H_1$ *.*

Proof. Follows from definition of φ and monomial order $>_{\mathcal{M}}$. □

Lemma 5.5. *If* $H = \underline{c_1 \mathbb{X}^{\alpha} \Psi(b\,,\,i)} + \sum_l c_l H_l$ *is a relation then* $\mathbb{X}^{\alpha} \neq X_0^{\alpha_0}$ *.*

Proof. If possible assume that $H = \underline{c_1 X_0^{\alpha_0} \Psi(b\,,\,i)} + \sum_l c_l H_l$. According to Lemma 5.4., there exist

l' such that $\mathrm{LM}(\varphi(H_{l'})) = \mathrm{LM}(\varphi(X_0^{\alpha_0}\Psi(b\,,\,i))) = X_0^{\alpha_0} X_p^a X_{b+i} t$. It is clear from the explicit

description of $\mathrm{LT}(\mathcal{G}')$, that no such $H_{l'}$ exist. Hence the proof. □

Lemma 5.6. *If* $H = \underline{c_1 \mathbb{X}^{\alpha} \Phi(i\,,\,j)} + \sum_l c_l H_l$ *is a relation then* $\mathbb{X}^{\alpha} \neq X_0^{\alpha_0}$ *.*

Proof. Similar to Lemma 5.5. □

Lemma 5.7. *If* $H = \underline{c_1 \mathbb{X}^{\alpha} \Psi(b\,,\,p-b)} + \sum_l c_l H_l$ *is a relation then*

$\mathbb{X}^{\alpha} \notin \{X_i^{\alpha_i} X_j^q \,:\, j \in [0,p]$ and $i \in \{0\,,\,p\}$ and $q \in \{0,1\}\,\}$.

Proof. If possible assume that $H = \underline{c_1 X_i^{\alpha_i} X_j^q \Psi(b\,,\,p-b)} + \sum_l c_l H_l$. According to Lemma 5.4.,

there exist l' such that $\mathrm{LM}(\varphi(H_{l'})) = \mathrm{LM}(\varphi(X_i^{\alpha_i} X_j^q \Psi(b\,,\,p{-}b))) = X_i^{\alpha_i} X_j^q X_p^{a+1} t$. From the explicit

description of $\mathrm{LT}(\mathcal{G}')$, it is clear that for $q = 0$ or, for $j \notin [b, p-1]$ no such $H_{l'}$ exist and for

$j \in [b, p-1]$ one can write $H_{l'} = X_i^{\alpha_i} X_p \Psi(b\,,\,j-b)$ but then $\mathrm{LM}(H) = H_{l'}$ will contradict the leading

monomial assumption. Hence the proof. $\square$

Lemma 5.8. *If* $H = \underline{c_1 \mathbb{X}^\alpha \Phi(i\,,\,j)} + \sum_l c_l H_l$ *is a relation then* $X_{l'} \mid \mathbb{X}^\alpha$ *for some* $0 < l' < j$.

Proof. If possible assume that $H = \underline{c_1 \mathbb{X}^\alpha \Phi(i\,,\,j)} + \sum_l c_l H_l$ and $X_{l'} \nmid \mathbb{X}^\alpha$ where $0 < l' < j$.

According to Lemma 5.4., there exist q such that $\mathrm{LM}(\varphi(H_q)) = \mathrm{LM}(\varphi(\mathbb{X}^\alpha \Phi(i\,,\,j))) = \mathbb{X}^\alpha X_i X_j t$.

From the explicit description of $\mathrm{LT}(\mathcal{G}')$, it is clear that $H_q = \begin{cases} \mathbb{X}^\beta \Phi(i\,,\,l) & \text{with} \quad l > j \\ \mathbb{X}^\beta \Phi(j\,,\,l) & \text{with} \quad l \geq j \\ \mathbb{X}^\beta \Phi(s\,,\,l) & \text{with} \quad s, l \geq j \end{cases}$

Therefore in all the cases $\mathrm{LM}(H) = H_q$. This contradiction proves the result. $\square$

6. Gröbner basis for the first syzygy module

For systematic reason set $\Phi(i\,,\,j) = \Phi(j\,,\,i)$ and $\Phi(i\,,\,j) = 0$ if $i, j \notin [1, p-1]$.

Let us construct the set $\widehat{\mathcal{G}}$ whose elements are the following(with underlined leading terms):

- $$A(i\,;\,b\,,\,j) = \underline{X_i \Psi(b\,,\,j)} - X_{b+i+j-\epsilon(i\,,\,b+j)} \Psi(b, \epsilon(i\,,\,b+j) - b) - X_p^a \Phi(i\,,\,b+j) + X_0^{a+d}[\Phi(i\,,\,j) - \Phi(b+i+j-p\,,\,p-b)]$$

- $B(i\,,\,j) = \underline{X_iX_j\Psi(b\,,\,p-b)} - X_{\epsilon(i\,,\,j)}X_{i+j-\epsilon(i\,,\,j)}\Psi(b\,,\,p-b) - \psi(b\,,\,p-b)\Phi(i\,,\,j)$
- $L(l\,;\,i\,,\,j) = \underline{X_l\Phi(i\,,\,j)} - X_j\Phi(i\,,\,l) + X_{\tau(i\,,\,j)}\Phi(i+j-\tau(i\,,\,j)\,,\,l) - X_{\tau(i\,,\,l)}\Phi(i+l-\tau(i\,,\,l)\,,\,j)$

Our aim is to prove that the set

$\widehat{\mathcal{G}} = \{A(i\,;\,b\,,\,j) \mid i \in [1,p] \quad \text{and} \quad j \in [0,p-b-1]\} \cup \{B(i\,,\,j) \mid i,j \in [1,p-1]\}$

$\cup \{L(l\,;\,i\,,\,j) \mid l,i,j \in [1,p-1] \quad \text{with} \quad i \leq j \quad \text{and} \quad l < j\}$

is a minimal Gröbner basis of $\mathcal{N}$. Note that $\widehat{\mathcal{G}} \subseteq \mathcal{N}$.

Theorem 6.1. *$\widehat{\mathcal{G}}$ is a Gröbner basis for the first Syzygy module of $\wp$ with respect to $>_{\mathcal{M}}$.*

Proof. If f is a monomial of $\mathcal{M}$ and $f \notin \mathrm{LT}(\widehat{\mathcal{G}})$ where,

$$\mathrm{LT}(\widehat{\mathcal{G}}) = \{X_iX_j\Psi(b\,,\,p-b) \mid i,j \in [1,p-1]\} \cup \{X_l\Phi(i\,,\,j) \mid l < j \text{ and } i \leq j\}$$
$$\cup \{X_l\Psi(b\,,\,j) \mid l \in [1,p] \text{ and } j \in [0,p-b-1]\}$$

then f must be one of the following :

Case(i) : $X_0^{\alpha_0}\Psi(b\,,\,i)$ Case(ii) : $X_0^{\alpha_0}X_i\Psi(b\,,\,p-b)$ Case(iii) : $X_p^{\alpha_p}X_i\Psi(b\,,\,p-b)$

Case(iv) : $\mathbb{X}^{\alpha}\Phi(i\,,\,j)$ with $X_l \nmid \mathbb{X}^{\alpha}$ for $0 < l < j$

Rest of the proof follows from Lemma 5.5. to Lemma 5.8. $\square$

Theorem 6.2. *The set $\widehat{\mathcal{G}}$ is a minimal Gröbner basis for the first Syzygy module of $\wp$.*

Proof. It is enough to note that no two distinct elements of $\mathrm{LT}(\widehat{\mathcal{G}})$ can divide each other. $\square$

Acknowledgements. I would like to express my heartfelt gratitude and sincere thanks to my supervisor Prof. Indranath Sengupta for his guidance, help and encouragement. I am also grateful and thankful to Prof. Tarun Kumar Mukherjee and Prof. Himadri Sarkar for their help and encouragement.

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Received: December, 2010