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# Periodic solutions of a quartic differential equation and Groebner bases

M.A.M. Alwash *

*Department of Mathematics, University of California, Los Angeles, CA 90024, USA*

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**Abstract**

We consider first-order ordinary differential equations with quartic nonlinearities. The aim is to find the maximum number of periodic solutions into which a given solution can bifurcate under perturbation of the coefficients. It is shown that this number is ten when the coefficients are certain cubic polynomials. Equations with the maximum number of such periodic solutions are also constructed. The paper is heavily dependent on computing Groebner bases.

*Keywords:* Nonlinear differential equations; Periodic solutions; Multiplicity; Bifurcation; Groebner bases; MAPLE

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## 1. Introduction

We consider differential equations of the form

$$\dot{z} = z^4 + \alpha(t)z^3 + \beta(t)z^2 + \gamma(t)z, \tag{1.1}$$

where $z$ is complex-valued; the coefficients are real polynomials in the real variable $t$. A solution $z(t)$ of (1.1) satisfying $z(0) = z(1)$ is called a periodic solution. The main concern is to estimate the number of periodic solutions. This problem was suggested by C. Pugh as a version of Hilbert's sixteenth problem (listed by S. Smale as Problem 7 in [6]). Recall that Hilbert's sixteenth problem is to determine the number of limit cycles of polynomial differential systems in the plane. In [3], Lins Neto gave examples to demonstrate that there is no upper bound for the number of periodic solutions unless suitable restrictions are placed on the coefficients. In his examples the coefficients are of degree $\leqslant 2n$ and the number of periodic solutions is $n+3$; the question raised in [3] is to construct equations with more than $n+3$ periodic solutions.

* E-mail: alwash@math.ucla.edu.


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The multiplicity of the periodic solution $\varphi(t)$ of (1.1) is defined to be the multiplicity of $\varphi(0)$ as a zero of the displacement function $q : c \mapsto z(1,c) - c$; $z(t,c)$ is the solution satisfying $z(0,c) = c$. A periodic solution is called a simple solution if its multiplicity is 1. In order to keep track of the number of periodic solutions, it is useful to take $z$ complex. This is because the number of zeros of $q$ in a bounded region of the complex plane cannot be changed by small perturbations. If the multiplicity of $\varphi(t)$ is $k$, then for any sufficiently small perturbations of the equation, there are precisely $k$ periodic solutions in a neighborhood of $\varphi(t)$ (counting multiplicity). This result is given in [4] in a more general form. Since the coefficients are real functions, an upper bound for the number of periodic solutions of (1.1) is also an upper bound for the corresponding equation with $z$ real. If the multiplicity is greater than 1 then, as was explained in [1], we could reduce Eq. (1.1) to the form

$$\dot{z} = z^4 + \alpha(t)z^3 + \beta(t)z^2. \tag{1.2}$$

This equation, with $z$ real, was considered in [5]; it was conjectured that the multiplicity of $z = 0$ is at most $n + 3$ if $\alpha$ and $\beta$ are polynomial functions in $t$ of degree $\leqslant n$. For the case $n = 2$, it is shown in [1] that the maximum multiplicity is 8. The method used in [1] to compute the multiplicity is mainly based on formulae derived to compute the multiplicity when it is $\leqslant 8$. The calculation of multiplicity for $n > 2$ is extremely difficult and time consuming.

In this paper, we explain how a computer algebra system can be used to compute the multiplicity and then to construct equations with many periodic solutions. The work which we describe depends on using the Groebner bases technique. We use **MAPLE** in computing Groebner bases. The method is applied for the case in which the coefficients are of degree 3; it is shown that the multiplicity of $z = 0$ is at most 10. Equations in this class are constructed with 10 periodic solutions. Thus, the upper bounds given in [3, 5] are exceeded.

Since we work in integer arithmetic, there are no rounding errors. It is certain that calculations by hand would be immeasurably slower and may be less reliable. However, special cases of the results presented here agree with those done by hand in [1].

The method in [1] of constructing equations with many periodic solutions is the bifurcations by successive perturbations. This method has to be modified in order to be applied to the cases considered in this paper. As explained in Section 4, this is done by using an exchange of stability argument.

In Section 2, a brief introduction to Groebner bases is given. The method of computing the multiplicity is described in Section 3, and the case $n = 3$ is then considered. In the last section, equations with many periodic solutions are constructed.

## 2. Groebner bases

The method of Groebner bases allows us to solve systems of polynomial equations in an algorithmic fashion. In this section, the basis concepts are presented. We restrict the discussion to the parts related to our work. The details can be found in [2].

Let $\boldsymbol{R}$ be the ring of all polynomials in $x_1, x_2, \ldots, x_n$ with real coefficients. A product $x_1^{m_1}x_2^{m_2}, \ldots, x_n^{m_n}$, with nonnegative exponents, is called a monomial. To define Groebner basis, we first have to fix a term order. The lexicographic order is used in our computations. This order is the most suitable to eliminate variables from a set of equations.

**Definition 2.1.** Let $x_n < x_{n-1} < \cdots < x_1$; then

$$x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n} < x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}$$

when the left-most nonzero number $k_i - m_i$ is positive.

Let $f = a_1p_1 + a_2p_2 + \cdots + a_mp_m$, with $a_i \neq 0$, constants, and $p_i$ are monomials satisfying $p_m < p_{m-1} < \cdots < p_1$. The leading term of $f$, written $\mathrm{lt}(f)$, is $a_1p_1$. If $f_1, f_2, \ldots, f_s$ are polynomials then the ideal generated by these polynomials is denoted by $\langle f_1, f_2, \ldots, f_s\rangle$. For an ideal $I \subseteq \boldsymbol{R}$ denote by $\mathrm{lt}(I)$ the set of leading terms of elements of $I$, and by $\langle \mathrm{lt}(I)\rangle$ the ideal generated by the elements of $\mathrm{lt}(I)$.

**Definition 2.2.** A finite subset $G = \{g_1, g_2, \ldots, g_s\}$ of an ideal $I$ is said to be a Groebner basis if $\langle \mathrm{lt}(g_1), \mathrm{lt}(g_2) \ldots, \mathrm{lt}(g_s)\rangle = \langle \mathrm{lt}(I)\rangle$. A Groebner basis is called a reduced Groebner basis for an ideal $I$ if for any $g_i$, the coefficient of $\mathrm{lt}(g_i)$ is 1 and no monomial of $g_i$ lies in $\langle \mathrm{lt}(G - \{g_i\})\rangle$.

The main properties of Groebner bases are summarized in the following proposition.

**Proposition 2.3** (Buchberger [2]). (1) *Let I be a polynomial ideal. For a given monomial order, I has a unique reduced Groebner basis.*

(2) *Any Groebner basis for an ideal I is a basis for I.*

(3) *Let G be a Groebner basis of an ideal I and f a polynomial. The remainder on division of f by G does not depend on the ordering of the elements of G. Moreover, f is an element of I if and only if the remainder is zero.*

(4) *Let* $f_1, f_2, \ldots, f_m$ *be polynomials. If the reduced Groebner basis of* $\langle f_1, f_2, \ldots, f_m\rangle$ *is* $\langle 1\rangle$ *then the equations* $f_1 = \cdots = f_m = 0$ *have no solutions; if the basis is not* $\langle 1\rangle$ *then they must have a solution (may be complex).*

In 1965, Buchberger presented an algorithm to compute the Groebner basis of any given ideal. Many computer algebra systems implement a version of Buchberger's algorithm. These systems usually compute a reduced Groebner basis. In our computations, the computer algebra system **MAPLE** is used. The most commonly used commands in **MAPLE's** Groebner basis package are:

(1) gbasis($F$, $X$, termorder): $F$ is a list of polynomials, $X$ is a list of indeterminates. It computes the reduced Groebner basis of the ideal $\langle G\rangle$ with respect to the indeterminates $X$ and the given term ordering.

(2) normalf($p$, $F$, $X$, termorder): $p$ is a polynomial, $F$ and $X$ are as in (1). It computes the fully reduced form of $p$ with respect to the ideal $\langle F\rangle$, indeterminates $X$ and the given term ordering.

We also used the commands *sturm* to find the number of real zeros of a polynomial, and *resultant* to find the resultant of two polynomials.

## 3. The calculation of multiplicity

Consider the differential equation

$$\dot{z} = z^4 + \alpha(t)z^3 + \beta(t)z^2. \tag{3.1}$$

Let $z(t,c)$ be the solution of (3.1) satisfying $z(0,c)=c$. For $0 \leqslant t \leqslant 1$ and $c$ in a neighborhood of 0, we write

$$z(t,c)=\sum_{n=1}^{\infty} a_n(t)\,c^n, \tag{3.2}$$

where

$$a_1(0)=1; \qquad a_n(0)=0 \quad (n>1). \tag{3.3}$$

Thus, the displacement function becomes

$$q(c)=(a_1(1)-1)c+\sum_{n=2}^{\infty} a_n(1)\,c^n.$$

The multiplicity of $z=0$ is $k>1$ if $a_1(1)=1$, $a_2(1)=a_3(1)=\cdots=a_{k-1}(1)=0$ and $a_k(1)\neq 0$. The functions $a_n(t)$ can be determined by substituting (3.2) into Eq. (3.1) and then comparing the coefficients of $c$. It is clear that $\dot{a}_1(t)=0$, which implies that $a_1(t)=1$. For $n>1$, the functions $a_n(t)$ are determined by the relation

$$\dot{a}_n=\sum_{i+j+k+l=n} a_i\,a_j\,a_k\,a_l+\alpha\sum_{i+j+k=n} a_i\,a_j\,a_k+\beta\sum_{i+j=n} a_i\,a_j. \tag{3.4}$$

These equations can be solved recursively. However, the necessary calculations become extremely complicated as $n$ increases. From (3.4), it follows that $\dot{a}_2=\beta, \dot{a}_3=\alpha+\beta a_2$. Therefore, the multiplicity of $z=0$ is 2 if $\int_0^1 \beta(t)\,\mathrm{d}t \neq 0$ and it is 3 if $\int_0^1 \beta(t)=0$ but $\int_0^1 \alpha(t)\neq 0$. The formulae for $a_n$, with $n \leqslant 8$ are given in [1]. If the coefficients are polynomials in $t$ then the functions $a_n(t)$ can be computed recursively using a computer algebra system.

Suppose that $\alpha$ and $\beta$ are polynomials in $t$. The first step is to compute the functions $a_n(t)$; these are polynomials in $t$ and the coefficients of $\alpha$ and $\beta$. **MAPLE** is used to compute $a_n(t)$ recursively from (3.4). Let $q_n=a_n(1)$. It is clear that $q_n$ are polynomial functions in the coefficients of $\alpha$ and $\beta$. To calculate the multiplicity, we reduce $q_n$ by means of substitutions from the relations $q_2=q_3=\cdots=q_{n-1}=0$. That is to compute the normal form of $q_n$ with respect to the Groebner basis of the ideal $\langle q_2,q_3,\cdots,q_{n-1}\rangle$. We continue in this procedure until the Groebner basis becomes $\langle 1\rangle$ or until we have a system with no real solutions. In the case the basis is $\langle 1\rangle$, we have to verify that the maximum multiplicity can be attained by certain real values of coefficients; since the unsolvability is over the field of complex numbers. The solution $z=0$ is an isolated periodic solution ([1, Theorem 2.2]); hence, the multiplicity of $z=0$ is finite.

Now we apply the above procedure to the case in which

$$\beta(t)=G+2At+3Bt^2, \qquad \alpha(t)=H+2Ct+3Dt^2+4Lt^3. \tag{3.5}$$

First consider the following particular cases.

**Lemma 3.1.** *Consider the differential equation* (3.1) *with $\alpha$ and $\beta$ as given in* (3.5).

(i) *Suppose that $L=0$, The multiplicity of $z=0$ is at most* 8; *there is a unique equation with multiplicity* 8.

(ii) *If $B=0$ then the multiplicity of $z=0$ is at most* 5.

**Proof.** (i) If the multiplicity is greater than 3 then $G = -A - B$ and $H = -C - D - L$. These values of $G$ and $H$ are substituted in the formulae of $q_n$, for $4 \leqslant n \leqslant 8$. The Groebner basis of $\langle q_4, q_5, q_6, q_7 \rangle$ is

$$\{D, -9B^2 + 110C, 3B + 2A, -2200 + 3B^3\}$$

and (the normal form)

$$q_8 = \frac{2888}{255\,255}B.$$

It follows that this set of equations has a unique solution and that $q_8 \neq 0$.

(ii) If $q_2 = q_3 = 0$ then $q_4 = \frac{1}{60}(AD + 2AL + 60)$. When $q_2 = q_3 = q_4 = 0$ then $q_5 = -\frac{1}{21}A$. The assumption $q_5 = 0$ contradicts $q_4 = 0$. Hence, the multiplicity is at most 5.

**Remark.** The result of Lemma 3.1(i) was obtained in [1], where the computations were done by hand.

We proceed with $B \neq 0$. The formula of $q_4$ is not linear in any of the coefficients. Hence, it is not possible to eliminate one of the variables from the assumption $q_4 = 0$. To make the computations easier, we introduce the change of variables $z \mapsto Bz$. Under this transformation Eq. (3.1) becomes of the form

$$\dot{z} = Bz^4 + \alpha(t)z^3 + \beta(t)z^2, \tag{3.6}$$

with

$$\beta(t) = G + 2At + 3t^2, \qquad \alpha(t) = H + 2Ct + 3Dt^2 + 4Lt^3. \tag{3.7}$$

It is clear that the transformation preserves the multiplicity and the number of periodic solutions.

**Theorem 3.2.** *If $\alpha(t)$ and $\beta(t)$ are as given in* (3.7) *with $B \neq 0$, then the multiplicity of $z = 0$ as a periodic solution of* (3.6) *is at most* 10. *There are exactly two equations with multiplicity* 9 *and only one equation with multiplicity* 10.

Before giving the proof of this theorem, we first need the following lemmas.

**Lemma 3.3.** *The following set of equations $g$ has exactly two solutions, and at these solutions $h \neq 0$.*

$$\begin{aligned} g = \{&L + 210B, C - L - DA - 2LA, 669 + 819A + 273A^2 + 91L, 690\,761\,659\,770D \\ &-529\,603\,074\,000AL^2 - 3\,784\,079\,710\,684LA - 143\,289\,911\,550A - 214\,934\,867\,325 \\ &-4\,294\,596\,246\,486L - 794\,404\,611\,000L^2, 356\,224\,811\,985 + 9\,658\,507\,581\,840L \\ &+49\,193\,036\,238\,892L^2 + 6\,884\,839\,962\,000L^3\}. \end{aligned}$$

$$\begin{aligned} h = &\,7\,432\,043\,813\,690\,284\,179\,819A + 2\,332\,460\,012\,229\,918\,021\,699 \\ &+9\,577\,421\,984\,298\,220\,973\,577A^2 + 6\,239\,817\,676\,615\,770\,003\,266A^3 \\ &+2\,055\,750\,840\,052\,527\,115\,680A^4 + 274\,100\,112\,007\,003\,615\,424A^5. \end{aligned}$$

**Proof.** The fifth polynomial in $g$ has three real roots; one root in each of the intervals $(-7,-6.9)$, $(-0.2,-0.15)$ and $(-0.1,-0.01)$. The third polynomial has real roots in $A$ only when $L$ takes the smallest value. For this value of $L$, there are two values for $A$. The values of $B, C$ and $D$ can be computed directly. The Groebner basis of the ideal $\langle g \cup \{h\}\rangle$ is $\langle 1\rangle$. Hence, $h \neq 0$ at the solutions of the set $g$.

**Lemma 3.4.** *The following set of equations $g$ has a unique solution and at this solution $h > 0$. Moreover, the values of $B$, $C$, and $L$ are positive*:

$$
\begin{aligned}
g = \{&2L + D, 9L + 420B - 7C, 534\,102\,076\,506\,106\,231\,200B \\
&-107\,327\,049\,014\,127\,655\,440L^4 + 49\,807\,865\,189\,040\,381\,894L \\
&-198\,097\,233\,956\,762\,834\,177L^2 + 259\,555\,217\,205\,378\,497\,832L^3 \\
&-4\,275\,888\,771\,408\,309\,720, 2\,404\,414\,264\,896L - 16\,621\,256\,838\,576L^2 \\
&+50\,374\,642\,156\,267L^3 - 127\,405\,376\,280 - 58\,739\,178\,025\,944L^4 \\
&+22\,837\,860\,143\,280L^5\},
\end{aligned}
$$

$$
\begin{aligned}
h = \;&4\,274\,756\,043\,356\,630\,109\,927\,956\,616\,625\,680L^4 \\
&+7\,888\,214\,947\,960\,280\,025\,283\,330\,278\,171\,769L^2 \\
&-10\,338\,106\,741\,515\,452\,011\,319\,376\,030\,658\,104L^3 \\
&+170\,673\,557\,528\,835\,516\,628\,724\,153\,391\,090 \\
&-1\,979\,593\,869\,697\,118\,470\,143\,039\,494\,712\,843L.
\end{aligned}
$$

**Proof.** First we consider the fourth polynomial in $g$. If we divide by the coefficient of $L^5$ then the coefficients in the resulting polynomial are contained in $(-3,3)$. Hence, the roots are contained in the interval $(-4,4)$. The real roots are positive. By Sturm test, it has only one real root, $l_o$, in this interval. In fact, the root is located in the interval $(0.13, 0.15)$. The other variables can be determined uniquely from the other polynomials. Using the same idea used in Lemma 3.3, $h'$ does not vanish in $(0.13, 0.15)$. The value of $h$ at 0.135 is positive. Hence, $h(l_o) > 0$. Similarly, the polynomials defining $B$ and $C$ are monotonic in $(0.13, 0.15)$ and positive at 0.135. Thus, the values of $B$ and $C$ are positive.

The Groebner basis, with respect to the list of variables $[B,C,D,A,L]$, of the ideal generated by $\{q_4, q_5, \ldots, q_n\}$ is denoted by $g_n$.

**Proof of Theorem 3.2.** If $q_2 = q_3 = 0$ then $G = -A - 1$ and $H = -C - D - L$. These values of $G$ and $H$ are substituted in the remaining $q_i's$. With this substitution, we have:

$$q_4 = \tfrac{1}{420}(9L - 7C + 420B + 7AD + 140AL).$$

If $q_4 = 0$ then

$$q_5 = -\tfrac{1}{8\,820}(2A+3)(L + 210B).$$

Therefore, $q_5 = 0$ when $2A + 3 = 0$ or $L + 210B = 0$. We consider these two cases separately.

(i) Case $L+210B=0$. The Groebner basis $g_8$ is the set of polynomials $g$ of Lemma 3.3, with the last three polynomials multiplied by $L$. If $L=0$ then $B=0$; this contradicts our assumption. The normal form of $q_9$ is $kh$, where $h$ is the one given in Lemma 3.3 and $k$ is a negative constant. Hence, by Lemma 3.3, the multiplicity is at most 9 with exactly two equations with this maximum multiplicity.

(ii) Case $2A+3=0$. The Groebner basis $g_8$ is the set $g$ given in Lemma 3.4, with the last three polynomials multiplied by $L$. The normal form of $q_9$ with respect to $g_8$ is zero; that is the multiplicity cannot be 9. The normal form of $q_{10}$ with respect to $g_8 = g_9$ is $kh$, where $h$ is given in Lemma 3.4 and $k$ is a positive constant. Therefore, by Lemma 3.4, the multiplicity is at most 10 with only one equation with this maximum multiplicity.

**Remark.** By the change of variable $z \mapsto B^{1/3}z$, Eq. (3.6) is transformed to the form (3.1). Therefore, the conclusions of Theorem 3.2 also hold for (3.1) with $\alpha$ and $\beta$ as in (3.5).

## 4. Bifurcation of periodic solutions

The method of constructing equations with many periodic solutions can be summarized in the following steps:

- Start with an equation for which the multiplicity of $z=0$ is $K$; that is $q_2=q_3=\cdots=q_{K-1}=0$ but $q_K \neq 0$. Let $U$ be a neighborhood of 0 in the complex plane containing no other periodic solution.
- Perturb the coefficients of the equation, if possible, so that the multiplicity is $K-1$. The total number of periodic solutions with initial values in $U$ is unchanged by sufficiently small perturbations. Hence, there is a nonzero periodic solution $\psi$, say, with $\psi(0)\varepsilon U$. Since complex solutions occur in conjugate pairs, it follows that $\psi$ is real. Let $W_1$ be a neighborhood of $\psi(0)$ and $U_1$ be a neighborhood of 0 such that $U_1 \cup W_1 \subset U$ and $U_1 \cap W_1 = \emptyset$. Make another perturbation so that the multiplicity of $z=0$ is $K-2$. A second real nonzero periodic solution with initial point in $U_1$ is bifurcated; there remains a real periodic solution with initial point in $W_1$.
- Continuing in this way, we end up with an equation with $K-2$ distinct nonzero periodic solutions and $z=0$ of multiplicity 2.

  This is the procedure used in [1] to construct equations with 8 periodic solutions. The same method cannot be used when $q_2=q_3=\cdots=q_{k-1}=0$ implying that $q_k=0$ for certain $k$. That is, the multiplicity cannot be $k$. In fact, this is the case for equations we are considering (as in Theorem 3.2). So the above steps fail to yield $K-2$ real periodic solutions. However, it is possible to bifurcate $K-2$ periodic solutions by exchanging the stability.
- Let $z=0$ be of multiplicity $k$, with $k$ even, and $q_k>0$. Hence, $z=0$ is unstable. Suppose that after a perturbation the multiplicity is $k-2$ and $q_{k-2}<0$; $z=0$ is stable. If the perturbation is sufficiently small, then two real nonzero periodic solutions bifurcate out of the origin; one is positive and the other is negative.

We use these steps to construct equations with 10 periodic solutions. Consider Eq. (3.6) with $\alpha$ and $\beta$ as in (3.7). To do the above procedure, the conditions which determine the multiplicity are needed. These conditions are obtained by computing the Groebner basis of $\langle q_2, q_3, \cdots, q_{n-1}\rangle$, and then computing the normal form of $q_n$ with respect to this basis. These are given in the following proposition (with $k_1, k_2, \cdots, k_6$ are positive real constants).

**Proposition 4.1.** *Suppose that $\alpha$ and $\beta$ are as in* (3.7) *with $G = -A - 1$ and $H = -C - D - L$. For the differential equation* (3.6), *the multiplicity of $z = 0$ is determined by*

(i) If $q_4 = k_1(420B - 7C + 9L + 7AD + 14LA) \neq 0$ then the multiplicity is 4.
(ii) If $q_4 = 0$ and $q_5 = -k_2(2A + 3)(L + 210B) \neq 0$ then the multiplicity is 5.
(iii) If the polynomials in the set $g_5$ vanish and $q_6 \neq 0$ then the multiplicity is 6:

$$g_5 = \{2A + 3\,840B - 14C - 21D - 24L\},$$

$$q_6 = -k_3(-21L + 52L^2 + 144\,144LB - 68\,796B + 50\,450\,400B^2) \neq 0.$$

(iv) If the polynomials in the set $g_6$ vanish and $q_7 \neq 0$ then the multiplicity is 7:

$$g_6 = \{-24L + 840B - 7(2C + 3D), -819(2C + 3D) - 2\,838L + 64\,792L^2 \\ +360\,360L(2C + 3D) + 5\,005(2C + 3D)^2, 2A + 3\},$$
$$q_7 = k_4(D + 2L)(273(2C + 3D) + 946L).$$

(v) If the polynomials in the set $q_7$ vanish and $q_8 \neq 0$ then the multiplicity is 8:

$$g_7 = \{9L + 420B - 7C, 10\,010C^2 - 24\,024LC + 14\,378L^2 + 1\,038L - 819C, \\ 2L + D, 2A + 3\},$$
$$q_8 = k_5(378\,010\,776CL^2 - 379\,018\,536LC + 44\,940\,168C - 478\,131\,732L^3 \\ +480\,953\,707L^2 - 57\,080\,886L).$$

(vi) If $A = -\frac{3}{2}$, and the polynomials in the set $g = g_8 = g_9$ of Lemma 3.4 vanish then the multiplicity is 10 and the origin is unstable. In this case, $q_{10} = k_6 h$, where $h$ is the polynomial in Lemma 3.4.

Let $a_0 = -\frac{3}{2}$ and let $b_0, c_0, d_0, l_0$ be the unique solution of the system $g_8$. Before giving the main result of this section, we prove a lemma.

**Lemma 4.2.** *Let*

$$h(L, C) = 10\,010C^2 - 24\,024LC + 14\,378L^2 + 1\,038L - 819C.$$

(i) *If $L = l_1 = l_0 + \varepsilon$ then there is a unique number $C = c_1$, a function of $\varepsilon$, satisfying $h(l_1, c_1) = 0$ and $\lim_{\varepsilon \to 0} c_1 = c_0$.*
(ii) *The directional derivative of $q_8$ in the direction of $h(L, C)$ at $(l_0, c_0)$ is negative.*

**Proof.** (i) The polynomial $h$ is an element of $g_7$; so it vanishes at $(l_0, c_0)$. The result follows, by the implicit function theorem, provided that $\partial h/\partial C \neq 0$ and $\partial h/\partial L \neq 0$ at $(l_0, c_0)$. But, $\partial h/\partial C = 20\,020C - 24\,024L - 819$. Substituting the value of $C$ from the relation $\partial h/\partial C = 0$ into $h$ gives

$$h = \tfrac{1}{440}(-16\,016L^2 + 24\,288L - 7\,371).$$

Similarly, $\partial h/\partial L = 0$ implies that

$$h = \tfrac{1}{72\,072}(-2\,616\,796L^2 + 3\,968\,328L - 1\,203\,561).$$

The resultant of each of these polynomials and the polynomial in $g_8$ which determined $l_0$ is not zero. Hence, the partial derivatives do not vanish at $(l_0, c_0)$.

(ii) The directional derivative is given by

$$\begin{aligned} d &= \frac{\partial h}{\partial C}\frac{\partial q_8}{\partial L} - \frac{\partial h}{\partial L}\frac{\partial q_8}{\partial C} \\ &= k\,(61\,932\,618\,419\,401\,690\,655\,197\,932\,L^3 - 47\,472\,516\,684\,148\,927\,365\,488\,252\,L^2 \\ &\qquad +11\,943\,820\,370\,438\,090\,329\,888\,569\,L - 1\,031\,706\,386\,063\,122\,476\,266\,595 \\ &\qquad -25\,540\,136\,280\,225\,260\,051\,707\,440\,L^4), \end{aligned}$$

where $k$ is a positive constant. By Sturm theorem (on **MAPLE**), the polynomial $d$ has no real roots; it is negative at $L = 0$. Hence, the directional derivative is negative.

In the followings $a_0$, $b_0$, $c_0$, $d_0$, $l_0$ are as defined above and $c_1$ as defined by Lemma 4.2.

**Theorem 4.3.** *Let*

$$\beta(t) = G + 2At + 3t^2 \quad \textit{and} \quad \alpha(t) = H + 2Ct + 3Dt^2 + 4Lt^3$$

*with*

$$\begin{aligned} A &= a_0 + \varepsilon_4, \\ B &= \tfrac{1}{420}(7c_1 - 9l_0) - \tfrac{3}{140}\varepsilon_1, \\ C &= c_1 - 3\varepsilon_2 + 2\varepsilon_4(\varepsilon_2 + 3\varepsilon_3) + \varepsilon_5, \\ D &= d_0 - 2\varepsilon_1 + 2\varepsilon_2 - 8\varepsilon_3, \\ L &= l_0 + \varepsilon_1 + 7\varepsilon_3, \end{aligned}$$

*and* $G = -A - 1 + \varepsilon_7$, $H = -C - D - L + \varepsilon_6$.

(i) *If* $\varepsilon_k\,(2 \leqslant k \leqslant 7)$ *are nonzero with* $\varepsilon_1 > 0$ *and such that* $|\varepsilon_k|$ *is sufficiently small compared to* $|\varepsilon_{k-1}|$ *then Eq.* (3.6) *has nine distinct real periodic solutions.*

(ii) *If, in addition,* $\varepsilon_8$ *is small enough then the equation*

$$\dot{z} = Bz^4 + \alpha(t)z^3 + \beta(t)z^2 + \varepsilon_8 z$$

*has ten distinct real periodic solutions.*

**Proof.** (i) If $\varepsilon_k = 0$ for $k = 1,\ldots,7$ then $z = 0$ is unstable and of multiplicity 10. With $\varepsilon_1 \neq 0$ but $\varepsilon_k = 0\,(k > 1)$, let $L = l_0 + \varepsilon_1$, and $D = d_1 = d_0 - 2\varepsilon_1$. Choose $C = c_1$, the value determined in Lemma 4.2, and let $B = b_1 = \frac{1}{420}(7c_1 - 9l_0 - 9\varepsilon_1)$. The polynomials in $g_7$ vanish at these values of the coefficients. The polynomials in the sets $g_7 \cup \{q_8\}$ and $g_8$ have the same solution set. Hence, for the values of $(L, C)$ satisfying $g_7$, $q_8$ vanishes only at $(l_o, c_o)$; since $g_8$ has a unique solution. In particular, $q_8(l_1, c_1) \neq 0$. By Lemma 4.2(ii), the function $q_8$ is negative if $\varepsilon_1 > 0$ and positive if $\varepsilon_1 < 0$. Hence, if $\varepsilon_1 > 0$ and small enough then the multiplicity is 8 and $z = 0$ is stable. The stability of the origin is reversed by this perturbation. By the argument presented at the beginning of the section, two real nonzero periodic solutions, $\varphi_1$ and $\varphi_2$, have bifurcated out of the origin.

Now, we choose $\varepsilon_2 \neq 0$ and $\varepsilon_k = 0 (k > 2)$. In this case, we let $D = d_2 = d_1 + 2\varepsilon_2$, and $C = c_2 = c_1 - 3\varepsilon_2$. The conditions in $g_6$ are satisfied and $q_7 = 4k_4\varepsilon_2(273c_1 - 346l_1)$. Here, we use the fact that $d_1 + 2l_1 = 0$. The possibility $273c_1 - 346l_1 = 0$ together with $g_7$ imply that $l_1 = c_1 = 0$. Therefore, if $\varepsilon_2$ is nonzero then the multiplicity is 7. A real periodic has bifurcated from the origin. If the perturbation is small enough, there remain two periodic solutions in a neighborhood of $\varphi_1$ and $\varphi_2$. We make another perturbation, $L = l_2 = l_1 + 7\varepsilon_3$, $D = d_3 = d_2 - 8\varepsilon_3$. In this case, the conditions in $g_5$ are satisfied and $q_6 = 7k_3\varepsilon_3(144\,144b_1 + 104l_1 + 364\varepsilon_3 - 21)$. Here, we use the fact that $q_6(b_1, l_1) = 0$. A fourth periodic solution is bifurcated out of the origin. The next two steps are similar. If $A = a_o + \varepsilon_4$, and $C = c_2 + 2\varepsilon_4(\varepsilon_2 + 3\varepsilon_3)$, then $q_5 = -k_2\varepsilon_4(l_2 + 210b_1)$; if $C = c_4 = c_3 + \varepsilon_5$ then $q_4 = -7k_1\varepsilon_5$. Finally, we perturb $H$ and $G$ by $\varepsilon_6$ and $\varepsilon_7$, respectively.

From the previous steps, we end up with an equation having nine distinct periodic solutions (including $z = 0$).

(ii) For the equation in part(i) $z = 0$ is still of multiplicity 2. By adding a linear term $\varepsilon_8 z$, another periodic solution is bifurcated.

**Remark.** Using the equation above and making the change of variables $z \mapsto B^{1/3}z$, an equation of the form (1.1) with ten periodic solutions can be constructed; the degrees of $\beta$ and $\alpha$ are still 2 and 3, respectively.

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