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Finite Lattices and Lexicographic Gröbner Bases

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By means of combinatorics on finite distributive lattices, lexicographic quadratic Gröbner bases of certain kinds of subrings of an affine semigroup ring arising from a finite distributive lattice will be studied.

INTRODUCTION

Recently, quadratic Gröbner bases have been studied by many papers on combinatorial commutative algebra, e.g., [11–14]. One of the reasons why commutative algebraists are interested in quadratic Gröbner bases is because the existence of a quadratic Gröbner basis of a homogeneous ideal $I \subset A$, where A is a polynomial ring, guarantees that the homogeneous K-algebra $R = A/I$ is Koszul [1]. Here, we do not state the details about Koszul algebras. Note, however, that to show that a given homogeneous K-algebra R is Koszul is, in general, quite difficult unless R admits a quadratic Gröbner basis. We refer the reader to, e.g., [2] and [6] for fundamental information about Gröbner bases.

Let K be a field, L a finite lattice, and $K[\{x_\alpha\}_{\alpha \in L}]$ the polynomial ring over K. Consider the ideal

$$I_L = (x_\alpha x_\beta - x_{\alpha\wedge\beta} x_{\alpha\vee\beta} \colon \alpha, \beta \in L)$$

of $K[\{x_\alpha\}_{\alpha \in L}]$. The quotient algebra

$$\mathcal{R}_K[L] = K[\{x_\alpha\}_{\alpha \in L}]/I_L$$

is called the *Hibi ring* of L over K.

In the case where L is a distributive lattice, the third author has shown [9] that $\mathcal{R}_K[L]$ is an algebra with straightening laws ([3, Chapter 7], [7] and [10, Part III]). In particular, it follows that I_L has a quadratic Gröbner basis for any term order which selects, for any two incomparable elements $\alpha, \beta \in L$, the monomial $x_\alpha x_\beta$ as the initial term of $x_\alpha x_\beta - x_{\alpha\wedge\beta} x_{\alpha\vee\beta}$. Such a term order is, for example, given by a rank reverse lexicographic term order, that is, the reverse lexicographic term order induced by a total ordering of the variables satisfying $x_\alpha < x_\beta$ if $\text{rank}(\alpha) > \text{rank}(\beta)$. If we instead choose the lexicographic term order induced by the same total ordering of the variables, then very simple examples show that the corresponding Gröbner basis is, in general, *not* quadratic. However, a rank lexicographic term order has the advantage that its Gröbner basis is restricted to interesting subrings of a Hibi ring, for example, to *rank bounded* subrings of a Hibi ring. As a consequence we find that these subrings have quadratic Gröbner bases, and hence are Koszul, provided the whole Hibi ring has a rank lexicographic quadratic Gröbner basis.

The natural problem arises as how to classify all finite distributive lattices which possess rank lexicographic quadratic Gröbner bases. Such a classification seems to be rather complicated. In the latter half of Section 2, however, a complete classification of the finite simple planar distributive lattices whose Hibi rings have rank lexicographic quadratic Gröbner bases will be obtained. See Theorem 2.5. It turns out that these lattices are exactly the chain ladders. The concept of chain ladders first appeared in [5].

In the former half of Section 2, it will be proved that the so-called trivial Hibi rings, first considered in [8], possess lexicographic quadratic Gröbner bases. The proof depends essentially on the fact (Theorem 1.2) that the Segre product $R * S$ of a homogeneous K-algebra

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R which admits a lexicographic quadratic Gröbner basis and a polynomial ring S has a lexicographic quadratic Gröbner basis. Even though the lexicographic term order presented for a trivial Hibi ring is not necessarily compatible with the rank of its lattice, again the restriction technique enables us to find certain kinds of subrings, which we call *lexsegment* subrings, of a trivial Hibi ring which possess quadratic Gröbner bases. See Theorem 2.2.

1. Segre Products and Lexicographic Term Orders

In this section we show that the Segre product $R * S$ of a homogeneous K-algebra R which admits a lexicographic quadratic Gröbner basis and a polynomial ring S has a lexicographic quadratic Gröbner basis.

For the proof of the main result of this section we shall use the following simple and well-known

Lemma 1.1. *Let $R = K[x_1, \dots, x_n]$ be the polynomial ring, $I \subset R$ a graded ideal and G a finite subset of homogeneous elements of I. Given a term order $<$, there exists a unique monomial K-basis B of $R/(\mathrm{in}_<(G))$ (which we call a 'standard basis' with respect to $<$ and G). If B is a K-basis of R/I, then G is a Gröbner basis of I with respect to $<$.*

Proof. By definition of Gröbner bases there is an epimorphism of graded K-algebras $\Phi\colon R/(\mathrm{in}_<(G)) \to R/\mathrm{in}_<(I)$. On the other hand, there is an isomorphism of graded K-vector spaces $\Psi : R/\mathrm{in}_<(I) \to R/I$. Note that G is a Gröbner basis with respect to $<$ if and only if Φ is an isomorphism, which is the case if and only if $\Psi \circ \Phi$ is an isomorphism. The last condition is guaranteed by our hypothesis. □

Theorem 1.2. *Let R be a homogeneous K-algebra which admits a lexicographic quadratic Gröbner basis, and let $S = K[y_1, \dots, y_m]$ be the polynomial ring. Then the Segre product $R * S$ has a lexicographic quadratic Gröbner basis.*

Proof. Let $R = A/I$, where $A = K[x_1, \dots, x_n]$ is the polynomial ring. Let $C = K[\{z_{ij}\}_{i=1,\dots,n,\, j=1,\dots,m}]$ be the polynomial ring in $n \cdot m$ variables, and $\Phi\colon C \to R * S$ be the homomorphism defined by $\Phi(z_{ij}) = x_i y_j$. Then $C/\operatorname{Ker}\Phi \cong R * S$.

We now introduce the lexicographic term order $<_{\mathrm{lex}}$ on C induced by the following order of the variables:

$$z_{ij} <_{\mathrm{lex}} z_{kl} \quad\Longleftrightarrow\quad \text{either} \quad i > k \qquad \text{or} \qquad (i = k \quad \text{and} \quad j > l).$$

Given a homogeneous polynomial $f = \sum_{i \le k} a_{i,k} x_i x_k$ of A, and $1 \le j, l \le m$, we define the following homogeneous polynomial

$$f_{j,l} = \sum a_{i,k} z_{ij} z_{kl}$$

in C. Let $G = \{f^{(1)}, \dots, f^{(p)}\}$ be a quadratic Gröbner basis of I with respect to the lexicographic term order $<$ induced by $x_1 > \cdots > x_n$. Set

$$G_1 = \{f^{(i)}_{j,l} \colon m \ge j \ge l \ge 1, 1 \le i \le p\};$$
$$G_2 = \{z_{ij} z_{kl} - z_{il} z_{kj} \colon 1 \le i < k \le n, 1 \le j < l \le m\}.$$

We will prove that $G_1 \cup G_2$ is a Gröbner basis of $\operatorname{Ker}\Phi$ with respect to $<_{\mathrm{lex}}$. Let B be the set of all monomials

$$z_{i_1 j_1} \cdots z_{i_d j_d} \qquad \text{with} \qquad 1 \le i_1 \le \cdots \le i_d \le n \qquad \text{and} \qquad m \ge j_1 \ge \cdots \ge j_d \ge 1$$

such that $x_{i_1} \cdots x_{i_d}$ is a standard monomial with respect to $<$ and G.

It follows from the definition of the Segre product and from the fact that G is a Gröbner basis of I, that $\Phi(B)$ is a K-basis of $R * S$. Therefore the set B is a K-basis of $C/\operatorname{Ker}\Phi$. On the other hand, it is clear that $G_1 \cup G_2 \subset \operatorname{Ker}\Phi$. We will show that B is a K-basis of $C/(\mathrm{in}_{<_{\mathrm{lex}}}(G_1 \cup G_2))$. Then, by Lemma 1.1, we are done.

It is enough to show that for every monomial $w \notin (\mathrm{in}_{<_{\mathrm{lex}}}(G_1 \cup G_2))$, one has $w \in B$. Write $w = z_{i_1 j_1} z_{i_2 j_2} \cdots z_{i_d j_d}$ where we can assume that $i_1 \leq i_2 \leq \cdots \leq i_d$. Moreover, since $w \notin (\mathrm{in}_{<_{\mathrm{lex}}}(G_2))$, we can assume that $j_1 \geq j_2 \geq \cdots \geq j_d$. So, it remains to show that $x_{i_1} \cdots x_{i_d}$ is a standard monomial with respect to $<$ and G. Suppose that $x_{i_1} \cdots x_{i_d} \in (\mathrm{in}_<(G))$. Then $x_{i_1} \cdots x_{i_d}$ is divided by $\mathrm{in}(f)$ for some $f \in G$. Say $f = \sum_{i \leq k} a_{i,k} x_i x_k$ and $\mathrm{in}_<(f) = x_t x_s$, $t \leq s$. Then $z_{tj} z_{sl}$ divides w for some $j \geq l$. We claim that $\mathrm{in}_{<_{\mathrm{lex}}}(f_{jl}) = z_{tj} z_{sl}$. This will give us the desired contradiction.

In order to check the last claim, let $z_{ij} z_{kl} \neq z_{tj} z_{sl}$, $i \leq k$, be a term in f_{jl}. Since $x_i x_k$ occurs in f, one has $x_t x_s > x_i x_k$. Therefore $t \leq i$. If $t = i$, then $s < k$, thus $z_{sl} > z_{kl}$. If $t < i$, then $z_{tj} > z_{ij}$ and, since $t < k$, one has $z_{tj} > z_{kl}$. Thus $z_{tj} z_{sl} > z_{ij} z_{kl}$. This completes the proof of the theorem. □

The above proof is no longer valid when the second ring S is not a polynomial ring. For example, if $R = K[x_1, x_2, x_3]$ and $S = K[y_1, y_2, y_3, y_4]/(y_2^2, y_1 y_4 + y_2 y_3)$, then $R * S$ does not have a quadratic Gröbner basis with respect to the lexicographic term order $<_{\mathrm{lex}}$ introduced in the proof of Theorem 1.2.

Even though the following result is almost obvious, it will play an essential role in the next section.

PROPOSITION 1.3. *Let $R = K[x_1, \ldots, x_n]/I$ and $S = K[y_1, \ldots, y_m]/J$ be homogeneous K-algebras, and assume that R (resp. S) has a quadratic Gröbner basis with respect to the lexicographic term order induced by $x_1 > \cdots > x_n$ (resp. $y_1 > \cdots > y_m$). Then the tensor product $R \otimes S$ has a quadratic Gröbner basis with respect to a lexicographic term order induced by any ordering of x_i's and y_j's satisfying $x_1 > \cdots > x_n$ and $y_1 > \cdots > y_m$.*

PROOF. If G_1 (resp. G_2) is a lexicographic quadratic Gröbner basis of R (resp. S), then it follows immediately from the Buchberger criterion that $G_1 \cup G_2$ is a desired Gröbner basis of $R \otimes S$. □

2. SUBALGEBRAS OF HIBI RINGS

The purpose of this section is to show that certain subalgebras of Hibi rings [9] possess lexicographic quadratic Gröbner bases. We will apply the technique of the restriction of Gröbner bases discussed in, e.g., [4].

First of all, we recall the definition of Hibi rings. Let L be a finite lattice, $K[\{x_\alpha\}_{\alpha \in L}]$ the polynomial ring over K and

$$I_L = (x_\alpha x_\beta - x_{\alpha \wedge \beta} x_{\alpha \vee \beta} : \alpha, \beta \in L)$$

the ideal of $K[\{x_\alpha\}_{\alpha \in L}]$. The quotient algebra

$$\mathcal{R}_K[L] = K[\{x_\alpha\}_{\alpha \in L}]/I_L$$

is called the *Hibi ring* of L over K. We are interested in Hibi rings associated with finite distributive lattices.

The fundamental structure theorem for finite distributive lattices guarantees that, for any finite distributive lattice D, there exists a unique finite poset (partially ordered set) P such that D is isomorphic to the lattice $\mathcal{J}(P)$ consisting of all poset ideals of P, ordered by inclusion. Here, a *poset ideal* of P is a subset Q of P (possibly empty) such that $\alpha \in Q$ and $\beta \in P$ together with $\beta \leq \alpha$ in P imply $\beta \in Q$.

Let $D = \mathcal{J}(P)$ be a finite distributive lattice with $P = \{x_1, x_2, \ldots, x_d\}$, and let $K[t, x_1, x_2, \ldots, x_d]$ be the polynomial ring over K. If $\alpha \in \mathcal{J}(P)$, then we write u_α for the squarefree monomial $t \prod_{x_i \in \alpha} x_i$ in $K[t, x_1, x_2, \ldots, x_d]$. It then follows from [9] that the affine semigroup ring $K[\{u_\alpha\}_{\alpha \in \mathcal{J}(P)}]$ is isomorphic to $\mathcal{R}_K[D]$. In fact, the kernel of the canonical surjective homomorphism $K[\{x_\alpha\}_{\alpha \in D}] \to K[\{u_\alpha\}_{\alpha \in \mathcal{J}(P)}]$ defined by $x_\alpha \mapsto u_\alpha$ coincides with I_D.

If P has an element p which is comparable with any element of P and if $P_1 = \{x \in P : x < p\}$ and $P_2 = \{x \in P : x > p\}$, then P is the ordinal sum [15, p. 100] $P_1 \oplus \{p\} \oplus P_2$ and

$$\mathcal{R}_K[\mathcal{J}(P)] = \mathcal{R}_K[\mathcal{J}(P_1)] \otimes \mathcal{R}_K[\mathcal{J}(P_2)].$$

If P is the direct sum [15, p. 100] $P_1 + P_2$, then

$$\mathcal{R}_K[\mathcal{J}(P)] = \mathcal{R}_K[\mathcal{J}(P_1)] * \mathcal{R}_K[\mathcal{J}(P_2)].$$

Let $\mathcal{O}$ denote the smallest class consisting of finite posets such that (i) any finite chain (totally ordered set) belongs to $\mathcal{O}$; (ii) if P and Q belong to $\mathcal{O}$, then the ordinal sum $P \oplus \{p\} \oplus Q$ belongs to $\mathcal{O}$, where $\{p\}$ is the chain consisting of one element; and (iii) if P belongs to $\mathcal{O}$ and if C is a finite chain, then the direct sum $P + C$ belongs to $\mathcal{O}$.

A Hibi ring $\mathcal{R}_K[D]$ with $D = \mathcal{J}(P)$ is called *trivial* if P belongs to $\mathcal{O}$. Note that the definition of trivial Hibi rings in the present paper is slightly different from that in [8].

A *chain decomposition* of a finite poset P is a set-theoretic decomposition $P = C_1 \cup C_2 \cup \cdots \cup C_s$, where each C_i is a chain of P and where $C_i \cap C_j = \emptyset$ for all $i \neq j$. A *canonical chain decomposition* of a finite poset P belonging to $\mathcal{O}$ is defined as follows. If C is a finite chain, then C itself is a canonical chain decomposition of C. If P and Q belong to $\mathcal{O}$ with canonical chain decompositions $P = C_1 \cup C_2 \cup \cdots \cup C_s$ and $Q = C'_1 \cup C'_2 \cup \cdots \cup C'_t$, then $C''_1 \cup C''_2 \cup C''_3 \cup \cdots$ with $C''_1 = C_1 \cup \{p\} \cup C'_1$ and $C''_i = C_i \cup C'_i$ for each $i \geq 2$, where $C_i = \emptyset$ if $i > s$ and $C'_j = \emptyset$ if $j > t$, is a canonical chain decomposition of the ordinal sum $P \oplus \{p\} \oplus Q$. If P belongs to $\mathcal{O}$ with a canonical chain decomposition $P = C_1 \cup C_2 \cup \cdots \cup C_s$ and if C is a finite chain, then $C_1 \cup C_2 \cup \cdots \cup C_s \cup C$ is a canonical chain decomposition of the direct sum $P + C$.

Let P be a finite poset belonging to $\mathcal{O}$ and fix a canonical chain decomposition $C_1 \cup C_2 \cup \cdots \cup C_s$ of P. We associate each poset ideal α of P with the sequence

$$\ell(\alpha) = (\sharp(\alpha \cap C_1), \sharp(\alpha \cap C_2), \ldots, \sharp(\alpha \cap C_s)) \in \mathbb{Z}^s,$$

where $\sharp(\alpha \cap C_i)$ is the number of elements of $\alpha \cap C_i$ for all i. See [15, pp. 111 and 112]. It follows easily that if α and β belong to $D = \mathcal{J}(P)$ with $\ell(\alpha) = (a_1, a_2, \ldots, a_s)$ and $\ell(\beta) = (b_1, b_2, \ldots, b_s)$, then $\ell(\alpha \wedge \beta) = (\min\{a_1, b_1\}, \min\{a_2, b_2\}, \ldots, \min\{a_s, b_s\})$ and $\ell(\alpha \vee \beta) = (\max\{a_1, b_1\}, \max\{a_2, b_2\}, \ldots, \max\{a_s, b_s\})$. The total ordering on the variables u_α, $\alpha \in D = \mathcal{J}(P)$, arising from the canonical chain decomposition $C_1 \cup C_2 \cup \cdots \cup C_s$ of P is the total ordering obtained by setting $u_\alpha < u_\beta$ if the left-most nonzero component of the vector difference $\ell(\alpha) - \ell(\beta)$ is positive. In particular, $u_\alpha < u_\beta$ if $\alpha > \beta$ in $D = \mathcal{J}(P)$.

Now, Theorem 1.2 and Proposition 1.3 yield the following

COROLLARY 2.1. *If P belongs to $\mathcal{O}$, then $\mathcal{R}_K[\mathcal{J}(P)]$ possesses a quadratic Gröbner basis with respect to the lexicographic term order induced by the total ordering arising from a canonical chain decomposition of P.*

PROOF. First, let P (resp. Q) belong to $\mathcal{O}$ and suppose that $\mathcal{R}_K[\mathcal{J}(P)]$ (resp. $\mathcal{R}_K[\mathcal{J}(Q)]$) possesses a quadratic Gröbner basis with respect to the lexicographic term order on $K[\{x_\alpha\}_{\alpha\in\mathcal{J}(P)}]$ (resp. $K[\{x_\alpha\}_{\alpha\in\mathcal{J}(Q)}]$) induced by the total ordering $<_1$ (resp. $<_2$) arising from a canonical chain decomposition $P = C_1 \cup C_2 \cup \cdots \cup C_s$ (resp. $Q = C_1' \cup C_2' \cup \cdots \cup C_t'$). We know by Proposition 1.3 that the tensor product $\mathcal{R}_K[\mathcal{J}(P)] \otimes \mathcal{R}_K[\mathcal{J}(Q)]$ possesses a quadratic Gröbner basis with respect to the lexicographic term order $<_{\text{lex}}$ on $K[\{x_\alpha\}_{\alpha\in\mathcal{J}(P)\oplus\mathcal{J}(Q)}]$ obtained by setting $x_\alpha <_{\text{lex}} x_\beta$ if (i) $\alpha, \beta \in \mathcal{J}(P)$ with $x_\alpha <_1 x_\beta$, or (ii) $\alpha, \beta \in \mathcal{J}(Q)$ with $x_\alpha <_2 x_\beta$, or (iii) $\alpha \in \mathcal{J}(Q)$ and $\beta \in \mathcal{J}(P)$. If we identify $\mathcal{J}(P) \oplus \mathcal{J}(Q)$ with $\mathcal{J}(P \oplus \{p\} \oplus Q)$ in the obvious way, then the lexicographic term order $<_{\text{lex}}$ coincides with the lexicographic term order induced by the total ordering arising from the canonical chain decomposition $C_1''\cup C_2''\cup C_3''\cup\cdots$ with $C_1'' = C_1\cup\{p\}\cup C_1'$ and $C_i'' = C_i \cup C_i'$ for each $i \geq 2$ of the ordinal sum $P \oplus \{p\} \oplus Q$.

Second, let P belong to $\mathcal{O}$ and let $<_1$ denote the total ordering arising from a canonical chain decomposition $C_1 \cup C_2 \cup \cdots \cup C_s$ of P. Suppose that $\mathcal{R}_K[\mathcal{J}(P)]$ possesses a quadratic Gröbner basis with respect to the lexicographic term order on $K[\{x_\alpha\}_{\alpha\in\mathcal{J}(P)}]$ induced by $<_1$. By virtue of Theorem 1.2 we know that the Segre product $\mathcal{R}_K[\mathcal{J}(P)] * S$ of $\mathcal{R}_K[\mathcal{J}(P)]$ and the polynomial ring $S = K[x_0, x_1, \ldots, x_m]$ possesses a quadratic Gröbner basis with respect to the lexicographic term order $<_{\text{lex}}$ on the polynomial ring $K[\{z_{(\alpha,i)}\}_{\alpha\in\mathcal{J}(P);0\leq i\leq m}]$ obtained by setting $z_{(\alpha,i)} <_{\text{lex}} z_{(\beta,j)}$ if and only if either $x_\alpha <_1 x_\beta$ or ($\alpha = \beta$ and $i > j$). Let $C: y_1 < y_2 < \cdots < y_m$ be any finite chain and $\gamma_i = \{y_1, y_2, \ldots, y_i\}$ with $\gamma_0 = \emptyset$. Then $\mathcal{J}(P + C) = \{\alpha \cup \gamma_i : \alpha \in \mathcal{J}(P), 0 \leq i \leq m\}$. The total ordering $<$ arising from the canonical chain decomposition $C_1 \cup C_2 \cup \cdots \cup C_s \cup C$ of $P + C$ is, by definition, obtained by setting $z_{(\alpha\cup\gamma_i)} < z_{(\beta\cup\gamma_j)}$ if and only if either $x_\alpha <_1 x_\beta$ or ($\alpha = \beta$ and $i > j$), since $\ell(\alpha \cup \gamma_i) = (\ell(\alpha), i) \in \mathbb{Z}^{s+1}$ for all $\alpha \cup \gamma_i \in \mathcal{J}(P + C)$. Hence, the trivial Hibi ring $\mathcal{R}_K[\mathcal{J}(P + C)]$ possesses the required quadratic Gröbner basis. □

Fix the total ordering $<$ arising from a canonical chain decomposition of P which belongs to $\mathcal{O}$. Let $\varphi, \psi \in D = \mathcal{J}(P)$ with $x_\varphi < x_\psi$ and write $\mathcal{R}_K[D]_\varphi^\psi$ for the subring of $\mathcal{R}_K[D]$ generated by all monomials u_α with $x_\varphi \leq x_\alpha \leq x_\psi$, i.e.,

$$\mathcal{R}_K[D]_\varphi^\psi = K[\{u_\alpha\}_{x_\varphi\leq x_\alpha\leq x_\psi}] = K[\{x_\alpha\}_{x_\varphi\leq x_\alpha\leq x_\psi}]/(I_D \cap K[\{x_\alpha\}_{x_\varphi\leq x_\alpha\leq x_\psi}]).$$

Such a subring of $\mathcal{R}_K[D]$ is called a *lexsegment* subring of $\mathcal{R}_K[D]$.

We are now in a position to state the first result of this section.

THEOREM 2.2. *All lexsegment subrings of a trivial Hibi ring possess lexicographic quadratic Gröbner bases.*

In order to prove Theorem 2.2 the following well-known technique will be required.

LEMMA 2.3. *Let A be a polynomial ring over a field K, I a homogeneous ideal of A, and S a subring of A generated by some of the indeterminates of A. Let $<$ be a term order of A, and G a Gröbner basis of I with respect to $<$ such that for all $f \in G$ with* $\text{in}_<(f) \in S$ *one has $f \in S$. Then $G \cap S$ is Gröbner basis of $I \cap S$.*

PROOF. Let $g \in I \cap S$; then, since $g \in I$ and G is a Gröbner basis of I, there exists $f \in G$ such that $\text{in}_<(f)$ divides $\text{in}_<(g)$. In particular, $\text{in}_<(f) \in S$. Thus our hypothesis implies that $f \in S$, and consequently, $f \in G \cap S$. This concludes the proof of the lemma. □

PROOF (PROOF OF THEOREM 2.2.). Let $\mathcal{R}_K[D]$ be a trivial Hibi ring, where $D = \mathcal{J}(P)$ and P belongs to $\mathcal{O}$. We know by Corollary 2.1 that I_D possesses a quadratic Gröbner basis, say G, with respect to the lexicographic term order induced by the total ordering $<$ arising from a canonical chain decomposition of P. Let $\mathcal{R}_K[D]_\varphi^\psi$ with $x_\varphi < x_\psi$ be a lexsegment subring of $\mathcal{R}_K[D]$.

Suppose that a quadratic binomial $x_\alpha x_\beta - x_\gamma x_\delta$ belongs to G with $x_\alpha x_\beta$ its initial part. Let $\ell(\alpha) = (a_1, a_2, \ldots)$, $\ell(\beta) = (b_1, b_2, \ldots)$, $\ell(\gamma) = (c_1, c_2, \ldots)$ and $\ell(\delta) = (d_1, d_2, \ldots)$. Then, $\min\{a_i, b_i\} = \min\{c_i, d_i\}$ and $\max\{a_i, b_i\} = \max\{c_i, d_i\}$ for all i, since $\alpha \wedge \beta = \gamma \wedge \delta$ and $\alpha \vee \beta = \gamma \vee \delta$. Let us assume that $x_\alpha < x_\beta$. Then, there is i with $a_j = b_j$ for all $j < i$ and with $a_i > b_i$. We next assume that $a_i = c_i$ and $b_i = d_i$. Thus, $x_\gamma < x_\delta$. Since $x_\alpha x_\beta$ is the initial part and since $x_\alpha < x_\beta$, we have $x_\delta < x_\beta$. Thus, there is $i' > i$ with $b_{j'} = d_{j'}$ for all $j' < i'$ and with $d_{i'} > b_{i'}$. Hence, $a_{j'} = c_{j'}$ for all $j' < i'$ and $a_{i'} > c_{i'}$. Thus, $x_\alpha < x_\gamma$. So, $x_\alpha < x_\gamma < x_\delta < x_\beta$.

Hence, if $x_\alpha x_\beta$ belongs to $K[\{x_\alpha\}_{x_\varphi \le x_\alpha \le x_\psi}]$, then $x_\alpha x_\beta - x_\gamma x_\delta$ must belong to $K[\{x_\alpha\}_{x_\varphi \le x_\alpha \le x_\psi}]$. Thus, by Lemma 2.3, the set $G \cap K[\{x_\alpha\}_{x_\varphi \le x_\alpha \le x_\psi}]$ is a Gröbner basis of $I_D \cap K[\{x_\alpha\}_{x_\varphi \le x_\alpha \le x_\psi}]$, as desired. $\square$

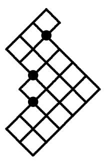
We now turn to a discussion of the existence of rank lexicographic quadratic Gröbner bases of finite distributive lattices. Let D be a finite distributive lattice. The *rank* of $\alpha \in D$ is the maximal integer k such that there exists a chain of D of the form $\alpha_0 < \alpha_1 < \cdots < \alpha_k = \alpha$, and is denoted by $\mathrm{rank}(\alpha)$. The rank of a unique maximal element of D is called the *rank* of D, and is denoted by $\mathrm{rank}(D)$. A *rank lexicographic* term order on $K[\{x_\alpha\}_{\alpha \in D}]$ is a lexicographic term order induced by a total ordering on D satisfying $\alpha < \beta$ if $\mathrm{rank}(\alpha) > \mathrm{rank}(\beta)$.

Fix integers $n < m$. The subring of $\mathcal{R}_K[D]$ generated by all monomials u_α with $n \le \mathrm{rank}(\alpha) \le m$ is called a *rank bounded* subring of $\mathcal{R}_K[D]$. It follows again from Lemma 2.3 that if a Hibi ring $\mathcal{R}_K[D]$ possesses a rank lexicographic quadratic Gröbner basis, then all rank bounded subrings of $\mathcal{R}_K[D]$ also possess lexicographic quadratic Gröbner bases. Thus, it is reasonable to ask which Hibi rings possess rank lexicographic quadratic Gröbner bases.

However, it seems difficult to find a combinatorial characterization for a finite distributive lattice to possess a rank lexicographic quadratic Gröbner basis. We give a solution of this classification problem for simple planar distributive lattices.

Let $\mathbb{N}^2$ denote the (infinite) distributive lattice consisting of all pairs (i, j) of nonnegative integers with the partial order $(i, j) \le (k, l) \iff i \le k, j \le l$. A *planar* distributive lattice is a finite sublattice D of $\mathbb{N}^2$ with $(0, 0) \in D$ such that, for any $(i, j), (k, l) \in D$ with $(i, j) < (k, l)$, there exists a chain of D of the form $(i, j) = (i_0, j_0) < (i_1, j_1) < \cdots < (i_s, j_s) = (k, l)$ such that $i_{k+1} + j_{k+1} = i_k + j_k + 1$ for all k. A planar distributive lattice D is called *simple* if, for all $0 < r < \mathrm{rank}(D)$, there exist at least two elements $\xi \in D$ with $\mathrm{rank}(\xi) = r$.

An element (i, j) of a simple planar distributive lattice D is said to be an *inner corner* of D if $(i-1, j), (i+1, j), (i, j+1)$ and $(i, j-1)$ belong to D and if either $(i+1, j-1) \notin D$ or $(i-1, j+1) \notin D$. A *chain ladder* (cf. [5]) is a simple planar distributive lattice D such that the set of all inner corners of D is a chain of D and that, for any two inner corners $(i, j) \neq (i', j')$ of D, one has $i \neq i'$ and $j \neq j'$. (See the figure below, where the dots denote the inner corners.)



The following combinatorial characterization of chain ladders might be of interest.

LEMMA 2.4. *A simple planar distributive lattice D is a chain ladder if and only if the following condition $(*)$ is satisfied:*

$(*)$ *If D possesses four elements $(i, j), (i, j'), (k, l)$ and (k', l) with $j < l < j'$ and $k < i < k'$, then either $(k, j') \in D$ or $(k', j) \in D$.*

PROOF. First, suppose that D fails to satisfy condition $(*)$ and choose four elements (i, j), (i, j'), (k, l) and (k', l) belonging to D with $j < l < j'$ and $k < i < k'$ such that $(k, j') \notin D$ and $(k', j) \notin D$. It then follows that there exist inner corners α and β of D with $(i, j) < \alpha < (k', l)$ and $(k, l) < \beta < (i, j')$. Note that α or β may be equal to (i, l), but $\alpha \neq \beta$, since D is simple. Hence, D cannot be a chain ladder.

Second, if D is not a chain ladder, then there exist two inner corners (i, j) and (k, l) with $(i, j) \neq (k, l)$ such that $i \leq k$ and $j \geq l$. Then, the four elements $(i, l-1)$, $(i-1, l)$, $(k+1, l)$ and $(i, j+1)$ belong to D. However, neither $(k+1, l-1)$ nor $(i-1, j+1)$ belong to D. Hence, D cannot satisfy condition $(*)$. $\square$

We now come to the second result of this section.

THEOREM 2.5. *Let D be a finite simple planar distributive lattice. Then $\mathcal{R}_K[D]$ has a rank lexicographic quadratic Gröbner basis if and only if D is a chain ladder.*

PROOF. We fix a rank lexicographic term order $<$ on $K[\{x_\alpha\}_{\alpha \in D}]$, and let G denote the set of all quadratic binomials $x_\alpha x_\beta - x_{\alpha\wedge\beta} x_{\alpha\vee\beta}$ such that $\alpha, \beta \in D$ are incomparable. Since D is a planar distributive lattice, $x_\alpha x_\beta \notin \mathrm{in}_<(I_D)$ if α and β are incomparable in D. Then, a monomial $x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_q}$ of $K[\{x_\alpha\}_{\alpha\in D}]$ is a standard monomial with respect to $<$ and G if and only if, for all $\alpha_i \neq \alpha_j$, either α_i and α_j are incomparable in D, or $\alpha_i < \alpha_j$ and $\alpha_i = \beta \wedge \gamma, \alpha_j = \beta \vee \gamma$ for *no* $\beta, \gamma \in D$. Let B denote the set of standard monomials with respect to $<$ and G.

We will show that B is a K-basis of $\mathcal{R}_K[D]$ if D is a chain ladder. We easily see that $\mathcal{R}_K[D]$ is spanned by B as a vector space over K. Thus, we must show that B is linearly independent. Let $(a, b) \in D$ be the unique minimal element of the inner corners of D and, without loss of generality, assume that all elements $(i, j) \in \mathbb{N}^2$ with $(i, j) \leq (a, b)$ belong to D. Suppose that B is linearly dependent and choose $w = x_{(i_1,j_1)} x_{(i_2,j_2)} \cdots x_{(i_q,j_q)}$ and $w' = x_{(i'_1,j'_1)} x_{(i'_2,j'_2)} \cdots x_{(i'_q,j'_q)}$ belonging to B with $w \neq w'$ such that the support of w coincides with that of w'. Here, the support of a monomial $w = x_{(i_1,j_1)} x_{(i_2,j_2)} \cdots x_{(i_q,j_q)}$ is the multichain $(a_1, b_1) \leq (a_2, b_2) \leq \cdots \leq (a_q, b_q)$ of D such that, as multisets, $\{i_1, i_2, \ldots, i_q\} = \{a_1, a_2, \ldots, a_q\}$ and $\{j_1, j_2, \ldots, j_q\} = \{b_1, b_2, \ldots, b_q\}$. Note that w is equal to w' in $\mathcal{R}_K[D]$ if and only if the support of w coincides with that of w'. Now, choosing the above $q \geq 2$ as small as possible, we may assume that $(i_s, j_s) \neq (i'_r, j'_r)$ for all s and r. Moreover, let us assume that $\min_{1\leq s\leq q} j_s = \min_{1\leq r\leq q} j'_r = 0$, say $j_1 = j'_1 = 0$ and $i_1 < i'_1 \leq a$. Then, for some $1 \leq s \leq q$, we have $i_s = i'_1$ and $j_s > 0$. The fact that D is a chain ladder guarantees that (i_1, j_s) belongs to D. Hence, w cannot be a standard monomial with respect to $<$ and G, a contradiction. This completes the proof of the 'if' part of the theorem.

In order to see why the 'only if' part of the theorem is true, suppose that a simple planar distributive lattice D is not a chain ladder. By Lemma 2.4 we can find four elements $(i, j), (i, j'), (k, l)$ and (k', l) belonging to D with $j < l < j'$ and $k < i < k'$ such that neither (k, j') nor (k', j) belong to D. Let $\alpha = (i, j)$, $\beta = (k, l)$, $\gamma = (i, j')$ and $\delta = (k', l)$. Then, $\alpha \vee \beta = \gamma \wedge \delta$. Let $w = x_{\alpha\wedge\beta} x_{\alpha\vee\beta} x_{\gamma\vee\delta}$. Then, both $x_\alpha x_\beta x_{\gamma\vee\delta}$ and $x_{\alpha\wedge\beta} x_\gamma x_\delta$ have the same support as w. Hence, for any rank lexicographic term order $<$ on $K[\{x_\alpha\}_{\alpha\in D}]$, either $x_\alpha x_\beta x_{\gamma\vee\delta}$ or $x_{\alpha\wedge\beta}x_\gamma x_\delta$ belong to $\mathrm{in}_<(I_D)$. However, none of $x_\alpha x_\beta$, $x_\alpha x_{\gamma\vee\delta}$, $x_\beta x_{\gamma\vee\delta}$, $x_{\alpha\wedge\beta}x_\gamma$, $x_{\alpha\wedge\beta}x_\delta$, $x_\gamma x_\delta$ belong to $\mathrm{in}_<(I_D)$. Hence, I_D cannot possess a rank lexicographic quadratic Gröbner basis, as required. □

REMARK 2.6. Even if D is a nonsimple planar distributive lattice, the above proof of Theorem 2.5 shows that $\mathcal{R}_K[D]$ has a rank lexicographic quadratic Gröbner basis if and only if D satisfies condition $(*)$ of Lemma 2.4. For example, if $P = \{\alpha, \beta\}$, where α and β are incomparable, and if D is the nonsimple planar distributive lattice $\mathcal{J}(P \oplus P)$, then $\mathcal{R}_K[D]$ has no rank lexicographic quadratic Gröbner basis.

By virtue of the proof of the ‘if’ part of Theorem 2.5 together with Lemma 2.3, we immediately obtain

COROLLARY 2.7. *Let D be a chain ladder and F a nonempty subset satisfying the condition as follows: if $\alpha \in D$ with $\alpha \leq \beta$ and $\alpha \geq \gamma$ for some $\beta, \gamma \in F$, then $\alpha \in F$. Then, the subring of $\mathcal{R}_K[D]$ generated by all monomials u_α with $\alpha \in F$ over K possesses a lexicographic quadratic Gröbner basis.*

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APPENDIX

We present here a quick introduction to Gröbner bases for combinatorialists. Let $A = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over a field K and write $\mathcal{M}(A)$ for the set of all monomials in A. Thus $\mathcal{M}(A)$ is a K-basis of A as a vector space over K. A *term order* on A is a total order $<$ on $\mathcal{M}(A)$ such that

(i) $1 < u$ for all $1 \neq u \in \mathcal{M}(A)$;
(ii) if $u < v$, then $uw < vw$ for all $w \in \mathcal{M}(A)$.

If $0 \neq f = c_1u_1 + \cdots + c_k u_m$ is a polynomial in A, where each $0 \neq c_k \in K$ and each $u_k \in \mathcal{M}(A)$, with $u_1 < u_2 < \cdots < u_m$, then the monomial u_m is said to be the *initial monomial* of f with respect to $<$ and is denoted by $\mathrm{in}_<(f)$. If $I \neq (0)$ is an ideal of A, then the *initial ideal* of I with respect to $<$ is the ideal of A generated by all monomials $\mathrm{in}_<(f)$ with $0 \neq f \in I$ and is written as $\mathrm{in}_<(I)$. A *Gröbner basis* of I with respect to $<$ is a finite set $G = \{g_1, \ldots, g_s\}$ of polynomials belonging to I such that $\mathrm{in}_<(I)$ is generated by $\mathrm{in}_<(g_1), \ldots, \mathrm{in}_<(g_s)$.

Dickson's lemma, which says that any nonempty subset of $\mathcal{M}(A)$ (in particular, $\mathrm{in}_<(I) \cap \mathcal{M}(A)$) has only finitely many minimal elements in the partial order by divisibility, guarantees that a Gröbner basis of I with respect to $<$ always exists. Moreover, it follows easily that if $G = \{g_1, \ldots, g_s\}$ is a Gröbner basis of I, then I is generated by $g_1, \ldots, g_s$.

A *lexicographic term order* on A induced by the total order $x_1 > x_2 > \cdots > x_n$ is the term order $<_{\mathrm{lex}}$ defined as follows: for monomials $u = x_1^{i_1} \cdots x_n^{i_n}$ and $v = x_1^{j_1} \cdots x_n^{j_n}$ in A with $u \neq v$, we set $u <_{\mathrm{lex}} v$ if $i_1 = j_1, \ldots, i_{k-1} = j_{k-1}$ and $i_k < j_k$ for some $1 \leq k \leq n$.

Let R be a homogeneous K-algebra and $R = A/I$, where I is a homogenous ideal of A. We say that R admits a quadratic Gröbner basis (resp. a lexicographic quadratic Gröbner basis) if there exists a term order $<$ on A (resp. a lexicographic term order $<_{\mathrm{lex}}$ on A induced by a total order of $x_1, \ldots, x_n$) such that the initial ideal of I with respect to $<$ (resp. $<_{\mathrm{lex}}$) is generated by quadratic monomials.

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