An Extension of the Extension Theorem

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Abstract. We propose an extended version of the classical Extension Theorem [2] to describe the image of an algebraic set $X \subset \mathbb{A}^n \times \mathbb{A}^1$ under the projection $\pi : \mathbb{A}^n \times \mathbb{A}^1 \to \mathbb{A}^n$. Furthermore, we apply the Extended Extension Theorem to compute the image of rational functions from $\mathbb{A}^1$ to $\mathbb{A}^n$ and the image of a projection of Whitney's umbrella.

1 Introduction

One of the many reasons why projective space $\mathbb{P}^n$ is in advantage to affine space $\mathbb{A}^n$ is the fact that $\mathbb{P}^n$ is a complete variety, i.e., for every variety V the projection map $\pi : V \times \mathbb{P}^n \to V$ is closed, i.e., maps closed sets onto closed sets. This is also known as the main theorem of elimination theory. A short and elegant proof of this is easily obtained from the valuative criterion of properness (of morphisms of schemes) [6]. However, as so often within the machinery of modern abstract algebraic geometry, the proof produces no equations of the image of a given closed subset. To obtain explicit equations, more elementary, constructive techniques, as resultants, lead to the aim. These techniques are quite old, however, for computer algebra indispensable. The main ingredient of the constructive proof (in [2]) of the main theorem of elimination theory is the so-called extension theorem [2] which, to a given closed subset $X \subset V \times \mathbb{A}^n$, gives an explicit, sufficient condition which of the points of the Zariski-closure $\overline{\pi(X)} \subset V$ are already in $\pi(X)$. More precisely, the extension theorem determines in a constructive way a closed subset W of V such that $\overline{\pi(X)} \setminus W \subset \pi(X)$. The motivation for the investigations of this paper was to find also a necessary condition, i.e., to find an explicit description of the set $\overline{\pi(X)} \setminus \pi(X)$ (and hence of $\pi(X)$). Since, unlike in the projective case, the set $\pi(X)$ generally isn't closed, it isn't enough to use only polynomial equations in the description of $\pi(X)$. A theorem of Chevalley [3], however, states that generally $\pi(X)$ is a constructible set, i.e., an element of the boolean lattice generated by the open (hence also closed) subsets of V in the power set of V. Thus, a constructible set is a finite union of locally closed subsets of V. Now, any locally closed subset of V can be described by finitely many equations $f = 0$ and not-equations $g \neq 0$, where f, g are regular (polynomial) functions on V. The main result of this paper is Theorem 2 which, for an algebraic set $X \subset \mathbb{A}^n \times \mathbb{A}^1$, explicitly describes the constructible set $\pi(X)$ by a simple formula consisting of finitely many equations and not-equations. The procedure of producing equations and not-equations which describe $\pi(X)$ from the equations describing the algebraic set X is also known as quantifier elimination (in the theory of algebraically closed fields). In the literature (cf. [7] [8] [9]) there are already fast algorithms doing this job, even for the general case of eliminating more than one variable. However, the descriptions of $\pi(X)$ are rather complicated and depend essentially on an algebraic procedure which decides the emptiness of algebraic sets. The advantage of our approach (in the special case of eliminating only one variable) is the simplicity and compactness of the description of the set $\pi(X)$. Moreover, the only cumbersome part in computing equations and not-equations describing the set $\pi(X)$ consists in computing Groebner bases of elimination ideals w.r.t. the projection π.

2 The Extended Extension Theorem

Let k be an algebraically closed field and $\mathbb{A}^n = \mathbb{A}^n(k)$ the affine space over k. In this section we fix a finite, non-empty set $\mathcal{F} = \{f_1, \dots, f_r\}$ of non-zero polynomials of $k[x_1, \dots, x_n, y]$ $(n \geq 1)$. Let $\pi : \mathbb{A}^n \times \mathbb{A}^1 \to \mathbb{A}^n$, $(x, y) \mapsto x$ be the projection map onto the first n coordinates. We consider the variety $X = V(\mathcal{F})$ defined by the polynomials of $\mathcal{F}$, and in the following we like to give an explicit description of the constructible set $\pi(X)$.

We consider polynomials $f \in k[x_1, \ldots, x_n, y] = k[x, y]$ as polynomials in the indeterminate y with coefficients in $k[x]$. Thus, if $f = \sum_{i=0}^{N} g_i(x) y^i$, $g_N \neq 0$, then $\deg_y(f) = N$, ($\deg_y(0) = -\infty$), $\mathrm{LC}_y(f) = g_N(x)$, $\mathrm{LT}_y(f) = g_N(x)y^N$ and $\ell_y(f) = |\{g_i : g_i \neq 0,\ i > 0\}|$.
We refer to [2][6] for standard notation and basic facts of algebraic geometry. As a non-standard notation we use $V_x(\mathcal{E}) = \{a \in \mathbb{A}^n : g(a) = 0 \text{ for all } g \in \mathcal{E}\}$ ($\mathcal{E} \subset k[x_1, \ldots, x_n]$) to distinguish with $V(\mathcal{E}) = \{(a, b) \in \mathbb{A}^n \times \mathbb{A}^1 : g(a) = 0 \text{ for all } g \in \mathcal{E}\}$ which equals $V_x(\mathcal{E}) \times \mathbb{A}^1$.

The essential idea to find an explicit description of $\pi(X)$ consists of a successive application of the classical extension theorem to a sequence of sets of polynomials iteratively generated from the initial set $\mathcal{F}$. Hence, we first recall the classical extension theorem (which is a slightly modified version of the extension theorem in [2]).

Theorem 1 (Extension Theorem). *Let $\mathcal{F}$, X and π as above. We define $\mathcal{E} = \{\mathrm{LC}_y(f) : f \in \mathcal{F},\ \deg_y(f) > 0\}$. If $\mathcal{E} = \emptyset$, then $\pi(X) = V_x(\mathcal{F})$ is closed. If $\mathcal{E} \neq \emptyset$, then $\overline{\pi(X)} \setminus V_x(\mathcal{E}) \subset \pi(X)$.*

Proof. Let $\tilde{\mathcal{E}} = \{\mathrm{LC}_y(f) : f \in \mathcal{F}\}$. Then, by the extension theorem in [2], $\overline{\pi(X)} \setminus V_x(\tilde{\mathcal{E}}) \subset \pi(X)$. If $\mathcal{E} = \emptyset$ then $\mathcal{F} \subset k[x_1, \ldots, x_n]$ and $X = V_x(\mathcal{F}) \times \mathbb{A}^1$, and thus $\pi(X) = V_x(\mathcal{F})$ is closed. Now, assume that $\mathcal{E} \neq \emptyset$. One easily verifies that $\overline{\pi(X)} \setminus V_x(\mathcal{E}) \subset \overline{\pi(X)} \setminus V_x(\tilde{\mathcal{E}})$, and we obtain also the second statement. □

Remark 1. Note that, with the notation of the proof of Theorem 1,

$$\overline{\pi(X)} \setminus V_x(\mathcal{E}) = \overline{\pi(X)} \setminus V_x(\tilde{\mathcal{E}})$$

holds since $\overline{\pi(X)} \subset V_x(\tilde{\mathcal{E}} \setminus \mathcal{E})$. Thus, considering the larger set $V_x(\mathcal{E})$ instead of the usually used $V_x(\tilde{\mathcal{E}})$ doesn't weaken the statement of the extension theorem (cf. [2]).

Definition 1. *Let $\mathcal{F}$, X and π as above. A sequence $\mathcal{F}_0, \ldots, \mathcal{F}_N$ of subsets of $k[x_1, \ldots, x_n, y]$ is called* admissible for $(\mathcal{F}, \pi)$ *if $\mathcal{F}_0 = \mathcal{F}$, $\mathcal{E}_i \neq \emptyset$ $(0 \leq i \leq N-1)$, $\mathcal{E}_N = \emptyset$ and*

$$V_x(\mathcal{E}_i) \cap \pi(X^i) = V_x(\mathcal{E}_i) \cap \pi(X^{i+1}) \qquad (0 \leq i \leq N-1)$$

where $\mathcal{E}_i = \{\mathrm{LC}_y(f) : f \in \mathcal{F}_i,\ \deg_y(f) > 0\}$ and $X^i = V(\mathcal{F}_i)$, $(0 \leq i \leq N)$.

Lemma 1. *Let $\mathcal{F}$ and π as above, let $N = \max\{\ell_y(f) : f \in \mathcal{F}\}$. We recursively define a sequence $\mathcal{F}_0, \ldots, \mathcal{F}_N$ of non-zero polynomials as follows:*

$$\mathcal{F}_0 = \mathcal{F}, \text{ and for } i = 1, \ldots, N$$
$$\mathcal{F}_i = \{f - \mathrm{LT}_y(f) : f \in \mathcal{F}_{i-1},\ \deg_y(f) > 0\} \cup \{g \in \mathcal{F}_{i-1} : \deg_y(g) \leq 0\}.$$

Then $\mathcal{F}_0, \ldots, \mathcal{F}_N$ is an admissible sequence for $(\mathcal{F}, \pi)$.

Proof. Let $\mathcal{E}_i$, X^i as in Definition 1. Due to the definition of N we have $\mathcal{E}_i \neq \emptyset$ $(0 \leq i \leq N-1)$ and $\mathcal{E}_N = \emptyset$. Assume $P \in \mathbb{A}^{n+1}$ is a point such that $\pi(P) \in V_x(\mathcal{E}_i)$. Then $\mathrm{LT}_y(f)(P) = 0$ for all $f \in \mathcal{F}_i$ with $\deg_y(f) > 0$. Hence, $P \in X^i$ iff $P \in X^{i+1}$. □

Theorem 2 (Extended Extension Theorem). *Let $\mathcal{F}$, X and π as above and let $\mathcal{F}_0, \ldots, \mathcal{F}_N$ be an admissible sequence for $(\mathcal{F}, \pi)$. Let $X^i = V(\mathcal{F}_i)$ and $\mathcal{E}_i = \{\mathrm{LC}_y(f) : f \in \mathcal{F}_i,\ \deg_y(f) > 0\}$. Assume $\mathcal{G}_i \subset k[x_1, \ldots, x_n]$ is a set of polynomials such that $V_x(\mathcal{G}_i) = \overline{\pi(X^i)}$, and let $\mathcal{D}_i = \mathcal{G}_0 \cup \bigcup_{j=1}^{i} \mathcal{G}_j \cup \mathcal{E}_{j-1}$, $0 \leq i \leq N$. Then $V_x(\mathcal{D}_i) \cap V_x(\mathcal{E}_i) \supset V_x(\mathcal{D}_{i+1})$ $(0 \leq i \leq N+1)$ and*

$$\pi(X) = V_x(\mathcal{D}_N) \cup \bigcup_{i=0}^{N-1} V_x(\mathcal{D}_i) \setminus V_x(\mathcal{E}_i).$$

To prove the theorem, we need the following lemma.

Lemma 2. *Let $\mathcal{D}_i$, $\mathcal{E}_i$, π and X^i be given as in Theorem 2. Then for* $0 \le i \le N-1$

(i) $V_x(\mathcal{D}_i \cup \mathcal{E}_i) \cap \pi(X^i) = V_x(\mathcal{D}_{i+1}) \cap \pi(X^{i+1})$
(ii) $V_x(\mathcal{D}_i) \setminus V_x(\mathcal{E}_i) \subset \pi(X^i)$
(iii) $V_x(\mathcal{D}_N) \cap \pi(X^N) = V_x(\mathcal{D}_N)$

Proof. (i) Since $\mathcal{F}_0, \ldots, \mathcal{F}_N$ is an admissible sequence for $(\mathcal{F}, \pi)$, we have

$$V_x(\mathcal{E}_i) \cap \pi(X^i) = V_x(\mathcal{E}_i) \cap \pi(X^{i+1}). \tag{1}$$

By construction $V_x(\mathcal{G}_{i+1}) = \overline{\pi(X^{i+1})}$. Hence, $V_x(\mathcal{D}_{i+1}) \cap \pi(X^{i+1}) = V_x(\mathcal{D}_i) \cap V_x(\mathcal{E}_i) \cap \pi(X^{i+1})$. This, together with (1), gives (i).

(ii) Since $i \neq N$, we have $\mathcal{E}_i \neq \emptyset$. Now, by Theorem 1, $\overline{\pi(X^i)} \setminus V_x(\mathcal{E}_i) \subset \pi(X^i)$. Since $V_x(\mathcal{D}_i)$ is a subset of $\overline{\pi(X^i)}$, we obtain (ii).

(iii) By construction, $V_x(\mathcal{G}_N) = \overline{\pi(X^N)}$. Since $X^N = V(\mathcal{F}_N)$ and $\mathcal{F}_N \subset k[x]$, we have $\overline{\pi(X^N)} = \pi(X^N)$. Thus $V_x(\mathcal{G}_N) = \pi(X^N)$, from which (iii) follows. □

Proof (of Theorem 2). If $N = 0$ then $\pi(X) = V_x(\mathcal{F}) = V_x(\mathcal{D}_N)$. Now, assume $N > 0$. We show per induction on $i = N, N-1, \ldots, 0$ that

$$V_x(\mathcal{D}_i) \cap \pi(X^i) = V_x(\mathcal{D}_N) \cup \bigcup_{j=i}^{N-1} V_x(\mathcal{D}_j) \setminus V_x(\mathcal{E}_j). \tag{2}$$

If $i = N$ then the assertion follows immediately from Lemma 2(iii). Now, assume $i < N$ and (2) is true for $i+1$. We write

$$V_x(\mathcal{D}_i) \cap \pi(X^i) = \left(V_x(\mathcal{D}_i) \setminus V_x(\mathcal{E}_i) \cap \pi(X^i)\right) \cup \left(V_x(\mathcal{D}_i \cup \mathcal{E}_i) \cap \pi(X^i)\right).$$

By applying Lemma 2(i) and (ii), we obtain

$$V_x(\mathcal{D}_i) \cap \pi(X^i) = V_x(\mathcal{D}_i) \setminus V_x(\mathcal{E}_i) \cup \left(V_x(\mathcal{D}_{i+1}) \cap \pi(X^{i+1})\right),$$

and the assertion follows immediately from the induction hypothesis. The theorem is proven if we have seen that $\pi(X) = V_x(\mathcal{D}_0) \cap \pi(X^0)$. This is, however, trivial since $X = X^0$ and $V_x(\mathcal{D}_0) = V_x(\mathcal{G}_0) = \overline{\pi(X^0)}$. □

Remark 2. Using Groebner bases it is easy to get finite sets $\mathcal{G}_i$ such that $V_x(\mathcal{G}_i) = \overline{\pi(X^i)}$ as required in the Extended Extension Theorem (cf., e.g., [2]). Thus Theorem 2 provides an algorithm to compute the constructible set $\pi(X)$.

3 Examples and Applications

We start with an easy example to illustrate the mechanism of the Extended Extension Theorem. The computations have been done with Singular Version 2.0.3 [4] [5].

Example 1. 1) Let $\mathcal{F} = \{xy - 1\}$. According to Lemma 1 $\mathcal{F}_0 = \mathcal{F}$, $\mathcal{F}_1 = \{-1\}$ is an admissible sequence. Now $\mathcal{E}_0 = \{x\}$, and let $\mathcal{G}_0 = \{0\}$, $\mathcal{G}_1 = \{1\}$. Then $\mathcal{D}_0 = \{0\}$ and $\mathcal{D}_1 = \{1, x\}$. Hence $\pi(V(xy-1)) = V_x(0) \setminus V_x(x) \cup V_x(1, x) = \mathbb{A}^1 \setminus \{0\}$.

2) Now, let $\mathcal{F} = \{xy\}$, and consider the admissible sequence $\mathcal{F}_0 = \mathcal{F}$, $\mathcal{F}_1 = \{0\}$. Then again $\mathcal{E}_0 = \{x\}$. Let $\mathcal{G}_0 = \mathcal{G}_1 = \{0\}$. We obtain $\mathcal{D}_0 = \{0\}$ and $\mathcal{D}_1 = \{0, x\}$. Thus $\pi(V(xy)) = V_x(0) \setminus V_x(x) \cup V_x(0, x) = \mathbb{A}^1$.

Example 2. Let $\mathcal{F} = \{f_1, f_2\}$, $f_1 = xt^2 - 2t + x$, $f_2 = (y-1)t^2 + y + 1$, and $\pi(t, x, y) = (x, y)$. Then $\mathcal{F}_0 = \mathcal{F}$, $\mathcal{F}_1 = \{2t + x, y + 1\}$, $\mathcal{F}_2 = \{x, y + 1\}$ is an admissible sequence for $(\mathcal{F}, \pi)$ due to Lemma 1.
We compute $\mathcal{E}_0 = \{x, y - 1\}$, $\mathcal{E}_1 = \{2\}$, and $\mathcal{G}_0 = \{x^2 + y^2 - 1\}$, $\mathcal{G}_1 = \{y + 1\}$, $\mathcal{G}_3 = \mathcal{F}_2 = \{x, y + 1\}$, and $\mathcal{D}_0 = \{x^2 + y^2 - 1\}$, $\mathcal{D}_1 = \mathcal{D}_0 \cup \{x, y - 1, y + 1\}$. Since $V_x(\mathcal{D}_1) = \emptyset$, we obtain $\pi(V(\mathcal{F})) = V_{(x,y)}(x^2 + y^2 - 1) \setminus \{(0, 1)\}$.

Remark 3. In example 2 we have computed the image of $t \mapsto (\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1})$, the parametrization of the unit-circle $\{x^2 + y^2 - 1 = 0\}$. We can use Theorem 2 to compute the image of any rational function from $\mathbb{A}^1$ to $\mathbb{A}^n$.

Proposition 1. *Let $F = (f_1/g_1, \ldots, f_n/g_n)$ be a rational function from $\mathbb{A}^1$ to $\mathbb{A}^n$, where $f_i = \sum_j c_{ij}t^j$ and $g_i = \sum_j d_{ij}t^j$ are relatively prime polynomials $(1 \le i \le n)$.*

(i) If $\deg(f_i) > \deg(g_i)$ for some i then $\mathrm{im}(F)$ is closed in $\mathbb{A}^n$.
(ii) Now assume $\deg(f_i) \le \deg(g_i)$ and $m_i = \deg(g_i) \ge 1$ $(1 \le i \le n)$. We set $c_{im_i} = 0$, if $\deg(f_i) < m_i$. Then $\overline{\mathrm{im}(F)} \setminus \mathrm{im}(F) \subset \{P\}$ with $P = (a_1, \ldots, a_n)$ and $a_i = d_{im_i}/c_{im_i}$.

Proof. Let π the projection $\pi(t, x, y) = (x, y)$ and I the ideal $I = (g_1x_1 - f_1, \ldots, g_nx_n - f_n)$, then $\mathrm{im}(F) = \pi(V(I))$. Let $\mathcal{E}$ be as in Theorem 1. In situation (i) $V_{(x,y)}(\mathcal{E})$ is empty. In (ii) $V_{(x,y)}(\mathcal{E}) = \{P\}$, and from Theorem 1 we obtain $\overline{\mathrm{im}(F)} \setminus \{P\} \subset \mathrm{im}(F)$. □

Remark 4. Proposition 1 was already shown by J.R. Sendra [10] (in the case $n = 2$). However, our proof is much simpler and not restricted to $n = 2$.

We obtain the following result about parametrizations of affine curves.

Corollary 1. *Assume $C \subset \mathbb{A}^n$ is an affine curve which admits a rational parametrization given by $F = (f_1/g_1, \ldots, f_n/g_n)$ where f_i, g_i are relatively prime $(1 \le i \le n)$. Then the parametrization covers the curve up to at most one point $P \in C$ Moreover, the coordinates of P are in the coefficient field of the polynomials f_i, g_i.*

Generally we can use Theorem 2 to compute the image of any rational function from $\mathbb{A}^1$ to $\mathbb{A}^n$. For if F is a rational function and P the point as in Proposition 1, then $\overline{\mathrm{im}(F)} = V(D_0)$ and $V(\mathcal{E}_0) = \{P\}$, where we have set $\mathcal{F} = \{g_1x_1 - f_1, \ldots, g_nx_n - f_n\}$. We only need to determine the largest $i \in \{0, \ldots, N\}$ such that $P \in V_{(x,y)}(\mathcal{D}_i)$. Note that $V_{(x,y)}(\mathcal{D}_i) \cap V_{(x,y)}(\mathcal{E}_i) \supset V_{(x,y)}(\mathcal{D}_{i+1})$, and hence for all $i > 0$ we have $P \in V_{(x,y)}(\mathcal{D}_i)$ iff $V_{(x,y)}(\mathcal{D}_i) \ne \emptyset$. If we have found such an i, we finally have to check if $P \in V_{(x,y)}(\mathcal{E}_i)$.

In the following examples we use Lemma 1 to compute an admissible sequence $(\mathcal{F}_i)$.

Example 3. Consider the tacnode curve $C \subset \mathbb{A}^2 = \mathbb{A}^2(\mathbb{C})$ given by the equation $2x^4 - 3x^2y + y^2 - 2y^3 + y^4 = 0$. The curve C admits a rational parametrization (cf. [11]) given by the rational function $F = (x(t), y(t))$

$$x(t) = \frac{t^3 - 6t^2 + 9t - 2}{2t^4 - 16t^3 + 40t^2 - 32t + 9} \qquad y(t) = \frac{t^2 - 4t + 4}{2t^4 - 16t^3 + 40t^2 - 32t + 9}$$

We compute $V_{(x,y)}(\mathcal{D}_0) = C$, $V_{(x,y)}(\mathcal{E}_0) = \{P\}$ where $P = (0, 0)$. Furthermore, $V_{(x,y)}(\mathcal{D}_1) = \{P\}$, and $V_{(x,y)}(\mathcal{E}_1) = V_{(x,y)}(\mathcal{D}_2) = \emptyset$. Hence, the parametrization covers the entire curve.

Example 4. The affine curve C given by the equation $(x^2 + 4y + y^2)^2 - 16(x^2 + y^2) = 0$ admits the following parametrization (cf. [11]):

$$x(t) = \frac{-1024t^3}{256t^4 + 32t^2 + 1} \qquad y(t) = \frac{-2048t^4 + 128t^2}{256t^4 + 32t^2 + 1}.$$

We compute $V_{(x,y)}(\mathcal{D}_0) = C$, $V_{(x,y)}(\mathcal{E}_0) = \{P\}$ with $P = (0, -8)$. Furthermore $V_{(x,y)}(\mathcal{D}_1) = \emptyset$. Hence we obtain $\mathrm{im}(F) = C \setminus \{P\}$. One easily checks that P lies on the curve C. Thus the above parametrization covers not the entire curve.

Remark 5. To decide whether $P = (P_1, \ldots, P_n) \in \mathrm{im}(F)$, where $F = (f_1/g_1, \ldots, f_n/g_n)$, we also could have proceeded (even simpler) as follows:
Compute $h = \gcd(g_1P_1 - f_1, \ldots, g_nP_n - f_n)$ and decide whether there is some $a \in \mathbb{A}^1$ such that $h(a) = 0$ and $g(a) \ne 0$, where $g = \prod_i g_i$. The latter can be done by computing the squarefree part h_{red} (cf. [1]) of h and testing whether h_{red} divides g, or not. Notice that generally a parametrization of a curve of genus 0 exists only over a quadratic extension of the base field [11]. Thus all computations have to be done in this quadratic extension field. Our intention to compute the above examples by using the extended extension theorem consists mainly in demonstrating the mechanism of the theorem.

Theorem 2 is not only applicable to parametrizations of (genus 0) curves, but also to parametrizations of "general kind" of curves, e.g., square-root parametrizations like $x^2 = f_1(t)/g_1(t)$, $y^2 = f_2(t)/g_2(t)$.

Definition 2. *A* parametrization of general kind *of an affine curve $C \subset \mathbb{A}^n$ is a rational function $F = (f_1/g_1, \ldots, f_n/g_n)$, where $f_i, g_i \in k[t]$ are relatively prime, together with polynomials $Q_1, \ldots, Q_n \in k[x_1, \ldots, x_n]$ such that $C = \overline{\mathrm{im}(F,Q)}$, where*

$$\mathrm{im}(F,Q) = \{x = (x_1, \ldots, x_n) \in \mathbb{A}^n : (Q_1(x), \ldots, Q_n(x)) \in \mathrm{im}(F)\} = Q^{-1}(\mathrm{im}\, F).$$

Proposition 2. *Assume (F,Q) is a parametrization of general kind of an affine curve $C \subset \mathbb{A}^n$, where $F = (f_1/g_1, \ldots, f_n/g_n)$, f_i, g_i relatively prime, and $Q_1, \ldots, Q_n \in k[x_1, \ldots, x_n]$.*

(i) If $\deg(f_i) > \deg(g_i)$ *for some i then* $\mathrm{im}(F,Q) = C$ *is closed in $\mathbb{A}^n$.*
(ii) If $\deg(f_i) \leq \deg(g_i)$, $\deg(g_i) \geq 1$ *for all i then $C \setminus \mathrm{im}(F,Q) \subset Q^{-1}(P)$, where P is as in Proposition 1.*

Proof. This follows immediately from Proposition 1 since Q is continuous. □

Example 5. Consider the affine curve C of genus 1 given by the equation $5x^2 - 4xy^2 - 2x + y^4 + y^2 - 1 = 0$. Then C admits the parametrization of general kind

$$x(t) = \frac{t^2 + t - 1}{t^2 + 1} \qquad y^2(t) = \frac{2t^2 - 3}{t^2 + 1}.$$

Let $(\mathcal{F}_i)$ be the admissible sequence from Lemma 1. Then we compute $V_{(x,y)}(\mathcal{D}_0) = C$, $V_{(x,y)}(\mathcal{E}_0) = \{(1, \pm\sqrt{2})\}$ and $V_{(x,y)}(\mathcal{D}_1) = \emptyset$. Hence we obtain $\mathrm{im}(P,Q) = C \setminus \{(1, \pm\sqrt{2})\}$.

Example 6. As our final example we compute the projection $\pi(X)$ of Whitney's umbrella $X = V(x^2z - y^2)$ where $\pi(x,y,z) = (x,y)$ – the projection is along the stick of the umbrella. Starting with the admissible sequence $\mathcal{F}_0 = \{x^2z - y^2\}$, $\mathcal{F}_1 = \{-y^2\}$, we obtain $V_{(x,y)}(\mathcal{D}_0) = \mathbb{A}^2$, $V_{(x,y)}(\mathcal{E}_0) = V_{(x,y)}(x^2) = \{0\} \times \mathbb{A}^1$ and $V_{(x,y)}(\mathcal{D}_1) = V_{(x,y)}(x^2, -y^2) = \{(0,0)\}$. Hence $\pi(X) = \mathbb{A}^2 \setminus (0 \times \mathbb{A}^1) \cup \{(0,0)\}$.

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