On Exact Solvability of Anharmonic Oscillators in Large Dimensions

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Abstract. General Schrödinger equation is considered with a central polynomial potential depending on $2q$ arbitrary coupling constants. Its exceptional solutions of the so called Magyari type (i.e., exact bound states proportional to a polynomial of degree N) are sought. In any spatial dimension $D \geq 1$, this problem leads to the Magyari's system of coupled polynomial constraints, and only purely numerical solutions seem available at a generic choice of q and N. Routinely, we solved the system by the construction of the Janet bases in a degree-reverse-lexicographical ordering, followed by their conversion into the pure lexicographical Gröbner bases. For very large D we discovered that (a) the determination of the "acceptable" (which means, real) energies becomes extremely facilitated in this language; (b) the resulting univariate "secular" polynomial proved to factorize, utterly unexpectedly, in a *fully non-numerical* manner. This means that due to the use of the Janet bases we found a new exactly solvable class of models in quantum mechanics.

1 Anharmonic Oscillators and the Problem of Their Solution

Elementary Hamiltonian $H = p^2 + q^2 + \lambda\, q^4$ of the so called anharmonic oscillator in one spatial dimension $D = 1$ is an example which plays a key role in quantum theory and in many of its applications. We may recollect, for illustration, that small experimental irregularities in the vibrational spectra in atomic physics are currently being attributed to the quartic anharmonicity at a suitable and, if possible, reasonably small coupling constant, $\lambda = \mathcal{O}(1)$ [1]. For the fit of some experimental data of this type one may even employ the two-parametric family of the Hamiltonians $H = p^2 + q^2 + \lambda\, q^4 + \varrho\, q^6$ [2], etc. In all these cases, sophisticated perturbation calculations are usually employed in order to achieve an agreement between experiment and theory (cf., again, ref. [1] and many other papers cited therein).

In a mathematically more ambitious setting, Magyari [3] was probably the first who noticed that in one dimension, Schrödinger equation admits non-perturbative, exceptional but exact bound-state solutions $\psi^{(Magyari)}(q)$ for *any* anharmonic potential of the following special polynomial form symmetric with respect to the origin,

$$V^{[q]}(r) = g_0\, r^2 + g_1\, r^4 + \ldots + g_{2q}\, r^{4q+2}\,, \qquad g_{2q} = \gamma^2 > 0\,, \tag{1}$$

provided only that its couplings g_j satisfy certain q constraints. These constraints have the form of the system of coupled polynomial equations (their form will be displayed and discussed below). Unfortunately, the achievement of the practical compatibility of the couplings with the Magyari's constraints requires the solution of his equations by a suitable more or less purely numerical technique. The corresponding algorithm is usually based on the use of Gröbner bases [4]. The procedure is very standard and one would have no particular reason for its study in more detail in general.

The first of the changes which proved relevant in this context appeared with the introduction of the higher-dimensional Schrödinger equations with polynomial interactions *and* with the Magyari-type solutions [5]. For all of these models, the Magyari-type equations become dependent on the

dimension $D \geq 1$ playing the role of a new formal parameter. The freedom in its choice will prove most relevant in our present paper but in a historical perspective, it still took many years before this chance has been conceived and described in ref. [6] where the choice of the potential proved restricted, for purely technical reasons, to the "first nontrivial" polynomial of type (1) with exponent $q = 2$.

The main inspiration of our present study of polynomial oscillators with $q > 2$ lies in the broad and not yet fully explored variety of the possibilities hidden in a consequent formal analysis of the Magyari-type equations. In this sense, the next decisive step has been made in refs. [7] where the Magyari's equations proved tractable in semi- and/or non-numerical manner at the first few lowest choices of the degree of the wave function, viz., at $N = 1$, $N = 2$, and $N = 3$. In these "trivial" cases, an overall tendency emerged of a distinct separation between the real (= physical) and complex (= apparently fully redundant) Magyari's couplings. This result offered an important hint for a more general analysis of the problem [8–10] and suggested the idea of using the Janet bases [11–13] in the similar cases. The simplicity of these low$-N$ models enabled us to see that an overall and more systematic study should be directed towards the domain of the very large $D \gg 1$ (cf. also refs. [14–17] in this respect).

In spite of the unique success of the *mathematics* of Magyari's nonlinear algebraic equations, a number of difficulties remained connected with their *practical* applications and applicability at the finite D. One of the key reasons (and differences from the harmonic oscillator and other exactly solvable models) is that the *explicit construction* of the Magyari's energies *remains purely numerical*. Indeed, these values (as well as the related couplings - we shall show some technical details below) must be computed as roots of a certain "secular" polynomial. This means that the difference between the variational, "generic $N = \infty$" rule in Hilbert space seems only marginally simplified by the Magyari-type construction of any $N \gg 1$ bound state.

The main purpose of refs. [6, 15] derived precisely from the latter point. Using the idea of perturbation expansions for Hamiltonians $H = H^{(q,N)}(D)$, these studies proceeded in two steps. Firstly, a zero-order approximations $H_0^{(q,N)}(\infty)$ have been constructed while, secondly, a series of corrections has been evaluated at each particular finite and fixed dimension $D < \infty$. This opened the market for the constructions of the Hamiltonians $H_0^{(q,N)}(\infty)$ in systematic manner.

In our present notation, the exact solvability of the zero-order Hamiltonians $H_0^{(q,N)}$ emerged as an utterly unexpected result of our calculations at $q = 1$ in ref. [15], at $q = 2$ in ref. [6] and at $q = 3$ in ref. [17]. In what follows we intend to address the next, more sophisticated problems with $q > 3$. An emphasis is to be put on the vital role of the methods which were able to produce the necessary final results *within the strict bound given by the not too fancy available computers*. Hence, in what follows, the main emphasis will be laid upon the quality of the underlying software. Still, a more detailed introductory chapter is due first.

2 The Derivation of the Magyari Equations

2.1 Harmonic Oscillator with $q = 0$ as a Methodical Guide

The partial differential Schrödinger equation for harmonic oscillator in D dimensions reads

$$\left(-\frac{\hbar^2}{2m}\triangle + \frac{1}{2}\, m\, \Omega^2\, |\boldsymbol{x}|^2\right) \Psi(\boldsymbol{x}) = \varepsilon\, \Psi(\boldsymbol{x}) \tag{2}$$

and is solvable by the separation of variables in several systems of coordinates. The most common cartesian choice may be recommended for the first few lowest spatial dimensions D only [18]. In contrast, the separation in spherical system remains equally transparent at any D because it reduces eq. (2) to *the same* ordinary (so called radial) differential equation

$$\left[-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + \omega^2 r^2\right]\, \psi(r) = E\, \psi(r) \tag{3}$$

with $r = |\boldsymbol{x}| \in (0, \infty)$, $E = 2m\varepsilon/\hbar^2$ and $\omega = m\Omega/\hbar > 0$. In this language we have $\ell = \ell_L = L + (D-3)/2$ where $L = 0, 1, \ldots$. At each L the energy levels are numbered by the second integer,

$$E = E_{n,L} = \omega\,(2n + \ell_L + 3/2), \qquad n, L = 0, 1, \ldots\ . \tag{4}$$

The wave functions with quadratic $\lambda(r) = \omega\, r^2/2 > 0$ and minimal $N = n + 1$ in

$$\psi_{n,L}(r) = r^{\ell+1}\, e^{-\lambda(r)} \sum_{m=0}^{N-1} h_m\, r^{2m} \tag{5}$$

are proportional to an nth Laguerre polynomial [19]. In Hilbert space, their set is complete.

2.2 $q = 1$ and Quasi-Exact (i.e., Incompletely Solvable) Sextic Oscillators

An immediate partially or quasi-exactly solvable (QES) generalization of harmonic oscillators was discovered by Singh et al [2]. In this case one replaces

$$\omega \longrightarrow W(r) = \alpha_0 + \alpha_1 r^2\,, \qquad V^{(HO)}(0) \longrightarrow G_{-1} + G_0 r^2 = U(r) \tag{6}$$

and gets the general sextic potential

$$V^{(sextic)}(r) = U(r) + r^2 W^2(r) = g_0\, r^2 + g_1\, r^4 + g_2\, r^6 \tag{7}$$

all three couplings of which are simple functions of our initial three parameters and *vice versa*,

$$g_2 = \alpha_1^2 > 0, \quad g_1 = 2\,\alpha_0\alpha_1, \quad g_0 = 2\,\alpha_0^2 + G_0, \quad G_{-1} = 0\,. \tag{8}$$

The resulting Schrödinger bound state problem cannot be solved in closed form. Nevertheless, we may postulate the polynomiality of the wave functions $\psi_{n,L}^{(sextic)}(r)$ *for a finite multiplet* (i.e., $N-$plet) of the wave functions. Under the specific constraint

$$G_0 = -\alpha_0^2 - \alpha_1(4N + 2\ell + 1), \qquad N \geq 1 \tag{9}$$

this $N-$plet of polynomial solutions (5) is made exact by the choice of a WKB-like (i.e., quartic) exponent

$$\lambda(r) = \frac{1}{2}\alpha_0 r^2 + \frac{1}{4}\alpha_1 r^4\,. \tag{10}$$

The ansatz (5) transforms then the differential Schrödinger equation into a linear algebraic definition of the unknown $N-$plet of coefficients h_m. The solution is always obtained for a mere finite set of the levels $n \in (n_0, n_1, \ldots, n_{N-1})$. In contrast to the harmonic oscillator, the QES solvability is based on the $L-$ and $N-$ dependent constraint (9) so that, generically, the elementary QES multiplet exists in a single partial wave only.

2.3 Magyari's QES Oscillators with $q > 1$

The explicit energy formula (4) for harmonic oscillator was replaced by an implicit definition in the preceding paragraph which gives the sextic QES energies *in the purely numerical form*, viz., as zeros of the Singh's secular determinant of a certain tridiagonal N by N matrix [20]. In this sense, Magyari [3] generalized the Singh's QES construction. In our present notation we may put, simply,

$$V^{(q)}(r) = U^{(q)}(r) + r^2[W^{(q)}(r)]^2\,, \qquad U^{(q)}(r) = G_0 r^2 + G_1 r^4 + \ldots + G_{q-1} r^{2q}\,,$$

$$W^{(q)}(r) = \alpha_0 + \alpha_1 r^2 + \ldots + \alpha_q\, r^{2q} \tag{11}$$

This formula re-parametrizes the polynomial (1) and specifies the one-to-one correspondence between the two sets of couplings,

$$\{g_0, \ldots, g_{2q}\} \Longleftrightarrow \{G_0, \ldots, G_{q-1}, \alpha_0, \ldots, \alpha_q\}$$

where $g_{2q} = {\alpha_q}^2$, $g_{2q-1} = g_{2q-1}(\alpha_q, \alpha_{q-1}) = 2\,\alpha_{q-1}\,\alpha_q, \ldots$ or, in opposite direction, $\alpha_q = \sqrt{g_{2q}} \equiv \gamma > 0$, $\alpha_{q-1} = g_{2q-1}/(2\alpha_q)$ etc.

At any $q = 1, 2, \ldots$, equation (10) must be further modified,

$$\lambda^{(q)}(r) = \frac{1}{2}\alpha_0 r^2 + \frac{1}{4}\alpha_1 r^4 + \ldots + \frac{1}{2q+2}\alpha_q r^{2q+2}\,. \tag{12}$$

With $\alpha_q > 0$, one verifies that

$$\psi^{(physical)}(r) \approx e^{-\lambda^{(q)}(r)+\mathcal{O}(1)}, \qquad r \gg 1$$

which means that the correct bound-state ansatz

$$\psi(r) = \sum_{n=0}^{N-1} h_n^{(N)}\, r^{2n+\ell+1} \exp\left[-\lambda^{(q)}(r)\right] \tag{13}$$

converts our radial equation (3) + (1) into an equivalent linear algebraic problem

$$\hat{Q}^{[N]}\, \boldsymbol{h}^{(N)} = 0 \tag{14}$$

with an asymmetric and *non-square* matrix

$$\hat{Q}^{[N]} = \begin{pmatrix}
B_0 & C_0 & & & & & & \\
A_1^{(1)} & B_1 & C_1 & & & & & \\
\vdots & & \ddots & \ddots & & & & \\
A_q^{(q)} & \ldots & A_q^{(1)} & B_q & C_q & & & \\
 & \ddots & & & \ddots & \ddots & & \\
 & & A_{N-2}^{(q)} & \cdots & A_{N-2}^{(1)} & B_{N-2} & C_{N-2} & \\
 & & & A_{N-1}^{(q)} & \cdots & A_{N-1}^{(1)} & B_{N-1} & \\
 & & & & \ddots & \vdots & \vdots & \\
 & & & & & A_{N+q-2}^{(q)} & A_{N+q-2}^{(q-1)} & \\
 & & & & & & A_{N+q-1}^{(q)} &
\end{pmatrix}. \tag{15}$$

Its elements depend on the parameters in bilinear manner,

$$\begin{gathered}
C_n = (2n+2)\,(2n+2\ell+3), \qquad B_n = E - \alpha_0\,(4n+2\ell+3) \\
A_n^{(1)} = -\alpha_1\,(4n+2\ell+1) + \alpha_0^2 - g_0, \qquad A_n^{(2)} = -\alpha_2\,(4n+2\ell-1) + 2\alpha_0\alpha_1 - g_1, \\
\ldots, \\
A_n^{(q)} = -\alpha_q\,(4n+2\ell+3-2q) + (\alpha_0\alpha_{q-1} + \alpha_1\alpha_{q-2} + \ldots + \alpha_{q-1}\alpha_0) - g_{q-1}, \\
n = 0, 1, \ldots\,.
\end{gathered} \tag{16}$$

At *any* fixed and finite $N = 1, 2, \ldots$ the non-square system (14) is an over-determined set of $N+q$ linear equations for the N non-vanishing components of the vector $\boldsymbol{h}^{(N)}$. At $q = 0$ these equations degenerate back to the recurrences and define the harmonic oscillator states. At $q = 1$ we return to the sextic model where the "redundant" last row fixes one of the couplings and where we are left with a diagonalization of an N by N matrix which defines the $N-$plet of the real QES energies in principle. The situation is more complicated at $q > 1$. The counting of parameters and equations indicates that unless one broadens the class of potentials, only a very small multiplet of bound states may remain available in closed form [21].

Using an elementary change of variables, one may transform the decadic forces into their quartic equivalents etc. Paper [22] may be consulted for details which indicate that the study of any potential $V(r)$ *which is a polynomial in any rational power of the coordinate* r may be replaced by the study of its present Magyari's or "canonical" QES representation $V^{(q)}(r)$ at a suitable integer q. In addition, we shall also restrict our attention to the domain of large D.

3 Magyari Equations at the Large Spatial Dimensions

Up to now, our attention has been concentrated upon the structure of the QES wave functions. From the point of view of the evaluation of the energies, the main dividing line between the solvable and unsolvable spectra is in fact marked by the distinction between the closed $q = 0$ formulae and their implicit QES form at $q = 1$. The transition to the next $q = 2$ may be perceived as merely technical. At all $q \geq 1$, the difficulties grow with N. In such a setting the emergence of certain simplifications at $D \gg 1$ may be crucial.

3.1 An Exceptional, Decoupled Last-Row Constraint

At any D, the last row in eq. (14) decouples from the rest of the system. At any $q > 1$ it may treated as a constraint which generalizes eq. (9),

$$g_{q-1} = -\alpha_q \,(4n + 2\ell + 3 - 2q) + (\alpha_0\alpha_{q-1} + \alpha_1\alpha_{q-2} + \ldots + \alpha_{q-1}\alpha_0)\,. \tag{17}$$

The insertion of this explicit definition of the coupling g_{q-1} simplifies the lowest diagonal in $\hat{Q}^{[N]}$,

$$A_n^{(q)} = 4\,\gamma\,(N + q - n - 1). \tag{18}$$

Since $A_{N+q-1}^{(q)} = 0$ we may drop the "hat" $\hat{}$ and re-write eq. (14) in the more compact form where the size of the non-square matrix $Q^{[N]}$ is merely $(N + q - 1)$ by N,

$$Q^{[N]}\,\boldsymbol{h}^{(N)} = 0\,. \tag{19}$$

This is the proper Magyari's system and it is merely solvable non-numerically in the simplest case with $q = 0$. No coupling is then fixed and the energies themselves are given by the explicit formula (17). Also the recurrences for coefficients of the wave functions may be solved in compact form.

The next, $q = 1$ version of eq. (19) degenerates to the single, determinantal secular equation

$$\det Q^{[N]} = 0. \tag{20}$$

Its solution is a purely numerical problem at all the larger $N \geq 5$. Of course, one coupling is fixed by eq. (17) and only the $N-$plet of energies must be calculated as represented by the real zeros of the single secular polynomial.

At the larger exponents $q \geq 2$, some q mutually coupled N by N secular determinants must vanish simultaneously [14]. With an auxiliary abbreviation for the energy $E = -g_{-1}$ this means that at least one of the couplings is always energy-dependent and that its value must be determined numerically. In the other words, our non-square matrix $Q^{[N]} = Q^{[N]}(g_{-1}, g_0, \ldots, g_{q-2})$ will annihilate the vector $\boldsymbol{h}^{(N)}$ if and only if all its q arguments are determined in a deeply nonlinear and self-consistent, mostly purely numerical manner.

3.2 Coupled Constraints at $D \gg 1$

In our approach the *guaranteed* polynomiality of the wave functions will play a key role. One can say that in our original differential eq. (3) the numerical value of the spatial dimension D will be assumed large. No other simplifications will be assumed.

In our problem with the old matrix elements

$$\begin{array}{c} C_n = (2n+2)\,(2n+2L+D), \qquad B_n = -g_{-1} - \alpha_0\,(4n+2L+D), \\ A_n^{(k)} = -g_{k-1} - \alpha_k\,(4n+2L+D-2k) + (\alpha_0\alpha_{k-1} + \ldots + \alpha_{k-1}\alpha_0)\,, \\ k = 1, 2, \ldots, q-1, \qquad n = 0, 1, \ldots, N+q-2 \end{array} \tag{21}$$

we shall preserve the dominant components of the matrix elements only,

$$C_n^{[0]} = (2n+2)\,D, \qquad B_n^{[0]} = -g_{-1} - \alpha_0\,D, \qquad A_n^{(k)[0]} = -g_{k-1} - \alpha_k\,D\ , k < q$$

(note that $A_n^{(q)} = A_n^{(q)[0]}$ is unchanged). Then we re-scale the coordinates and, hence, coefficients according to the rule

$$h_n^{(N)} = p_n/\mu^n\,. \tag{22}$$

Simultaneously we have to replace the energies and couplings $\{g_{-1}, g_0, \ldots, g_{q-2}\}$ by the new re-scaled parameters $\{s_1, s_2, \ldots, s_q\}$ using the following linear recipe,

$$g_{k-2} = -\alpha_{k-1} D - \frac{\tau}{\mu^{k-1}}\, s_k, \qquad k = 1, 2, \ldots, q\,. \tag{23}$$

where we abbreviated

$$\mu = \mu(D) = \left(\frac{D}{2\gamma}\right)^{1/(q+1)}, \qquad \tau = \tau(D) = \left(2^{q+2}\, D^q\, \gamma\right)^{1/(q+1)}. \tag{24}$$

In the leading-order approximation this gives, finally, our Magyari equations in the compact form

$$\begin{pmatrix}
s_1 & 1 & & & & & \\
s_2 & s_1 & 2 & & & & \\
\vdots & & \ddots & \ddots & & & \\
s_q & \vdots & & s_1 & N-2 & & \\
N-1 & s_q & & & s_1 & N-1 & \\
 & N-2 & s_q & & \vdots & s_1 & \\
 & & \ddots & \ddots & & \vdots & \\
 & & & 2 & s_q & s_{q-1} & \\
 & & & & 1 & s_q &
\end{pmatrix}
\begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{N-2} \\ p_{N-1} \end{pmatrix} = 0 \tag{25}$$

which is to be studied in what follows.

4 The Method of Solution of the $D \gg 1$ Magyari Equations

4.1 Involutive Bases

To solve polynomial systems (25) we shall construct for them the related Janet bases. Janet bases are typical representatives of general involutive bases of polynomial ideals [23] which are Gröbnerian though, generally, redundant. However, just this redundancy of involutive bases makes the structural and combinatorial information on polynomial and differential ideals and modules more accessible [24–27].

And as well as the reduced Gröbner bases, the involutive bases can be used for solving polynomial systems with finitely many solutions that correspond to the zero-dimensional ideals [28]. For this purpose, a pure lexicographical monomial order seems best since it provides the completely triangular basis with sequentially eliminated variables starting from the highest one with respect to the order chosen [4]. However, computation of a lexicographical basis takes usually much more time than computation of a degree-reverse-lexicographical basis, first, and conversion of this basis into the lexicographical one, second.

We use this two-step computational procedure in our study and solving (25). In doing so we shall deal with the minimal Janet bases [29] only. It is remarkable that a degree-reverse-lexicographical order is inherent in minimal Janet bases of zero-dimensional ideals. What follows from the demonstration in [26] is this inherence to Pommaret bases, and the fact proven in [30] that a minimal Janet basis is also a Pommaret basis whenever the latter exists, i.e., whenever it is finite. Zero-dimensional ideals always have finite Pommaret bases [31].

4.2 Janet Bases

Below we use the definitions and notations from [29, 30, 32, 33]: $\mathbb{N}$ is the set of non-negative integers; $\mathbb{M} = \{x_1^{d_1} \cdots x_n^{d_n} \mid d_i \in \mathbb{N}\}$ is the set of monomials in the polynomial ring $\mathbb{R} = \mathbb{K}[x_1, \ldots, x_n]$ over zero characteristic field $\mathbb{K}$; $\deg_i(u)$ is the degree of x_i in $u \in \mathbb{M}$; $\deg(u) = \sum_{i=1}^{m} \deg_i(u)$ is the total degree of u; $\succ$ is an admissible [4, 28] monomial ordering compatible with

$$x_1 \succ x_2 \succ \cdots \succ x_n \,.$$

Divisibility of monomial v by monomial u will be written as $u \mid v$. A divisor u of a monomial v is proper if $deg(u) < deg(v)$. $\mathrm{lm}(f)$ and $lt(f)$ denote, respectively, the leading monomial and the leading term of the polynomial $f \in \mathbb{R}$ with respect to $\succ$. $\mathrm{lm}(F)$ denotes the leading monomial set for F, and $Id(F)$ denotes the ideal in R generated by F.

Let polynomial set $F \subset \mathbb{R}$ be finite and $f \in F$. For each $1 \le i \le n$ we partition F into groups labeled by non-negative integers $d_1, \ldots, d_i$:

$$[d_1, \ldots, d_i] = \{\ f\ \in F \mid d_j = \deg_j(\mathrm{lm}(f)),\ 1 \le j \le i\ \}.$$

A variable x_i is *(Janet) multiplicative* for $f \in F$ if $i = 1$ and

$$\deg_1(\mathrm{lm}(f)) = \max\{\deg_1(\mathrm{lm}(g)) \mid g \in F\},$$

or if $i > 1$, $f \in [d_1, \ldots, d_{i-1}]$ and

$$\deg_i(\mathrm{lm}(f)) = \max\{\deg_i(\mathrm{lm}(g)) \mid g \in [d_1, \ldots, d_{i-1}]\}.$$

If a variable is not multiplicative for $f \in F$, it is *nonmultiplicative* for f and we write this as $x_i \in NM_J(f, F)$. $u \in \mathrm{lm}(F)$ is a *Janet divisor* of $w \in \mathbb{M}$, if $u \mid w$ and monomial w/u contains only multiplicative variables for u. In this case we write $u \mid_J w$.

A finite polynomial set F is *Janet autoreduced* if each term in every $f \in F$ has no Janet divisors among $\mathrm{lm}(F) \setminus \mathrm{lm}(f)$. A polynomial $h \in \mathbb{R}$ is in the *Janet normal form modulo* F if every term in h has no $J-$ divisors in $\mathrm{lm}(F)$. We denote the Janet normal form of polynomial f modulo F by $NF_J(f, F)$. If the leading monomial $\mathrm{lm}(f)$ of f has no Janet divisors among elements in $\mathrm{lm}(F)$, then we say that f is in the *Janet head normal form* modulo F and write $f = HNF_J(f, F)$.

A Janet autoreduced set F is a *Janet basis* of $Id(F)$ if any *nonmultiplicative prolongation* (multiplication by a nonmultiplicative variable) of any polynomial in F has vanishing Janet normal form modulo F:

$$(\forall f \in F)\ (\forall x \in NM_J(f, F))\ \ [\ NF_J(f \cdot x, F) = 0\ ]\,. \tag{26}$$

A Janet basis G of ideal $Id(G)$ is *minimal* if for any other Janet basis F of the ideal the inclusion $lm(G) \subseteq \mathrm{lm}(F)$ holds. A monic minimal Janet basis is uniquely defined by an ideal and a monomial order. In what follows we deal with the minimal Janet bases only and often omit the word "minimal".

4.3 Algorithm for Computing Janet Bases

We present now the algorithm **JanetBasis** which is a special form of the general Gerdt–Blinkov algorithm [9, 29] for computing minimal involutive bases concretized for Janet division. This concretization in its more detailed form relied on the appropriate data structures – Janet trees – and is described in [32, 33]. Note that, recently, the Gerdt–Blinkov algorithm in its form presented in [9, 25] was implemented in Maple for both the polynomial and the linear differential ideals [12, 13].

To provide minimality of the output Janet basis [29] the intermediate data, i.e. initial polynomials and their prolongations and reductions, are partitioned into two subsets T and Q. Set T contains a part of the intermediate basis. Another part of the intermediate data contained in set Q also includes all the nonmultiplicative prolongations of polynomials in T which must be examined in accordance with the definition of Janet bases.

To apply the involutive analogues of the Buchberger criteria and to avoid repeated prolongations we endow with every polynomial $f \in F$ the triple structure

$$p = \{f,\, u,\, vars\}$$

such that

$$\begin{aligned} \mathrm{pol}(p) &= f \text{ is polynomial } f \text{ itself,} \\ \mathrm{anc}(p) &= u \text{ is the leading monomial of a polynomial ancestor of } f \text{ in } F, \\ \mathrm{nmp}(p) &= vars \text{ is a (possible empty) subset of variables.} \end{aligned}$$

Here the *ancestor* of f is a polynomial $g \in F$ with $u = \mathrm{lm}(g)$ and such that $u \mid \mathrm{lm}(p)$. Moreover, if $\deg(u) < \deg(\mathrm{lm}(p))$, then every variable occurring in the monomial $\mathrm{lm}(p)/u$ is nonmultiplicative for g. Besides, for the ancestor g the equality $\mathrm{anc}(g) = \mathrm{lm}(g)$ must hold. These conditions mean that polynomial p was obtained from g, in the course of the below algorithm **JanetBasis**, by a sequence of nonmultiplicative prolongations. This tracking of the history in the algorithm allows one to use the involutive analogues of Buchberger's criteria to detect and avoid unnecessary reductions.

The set *vars* contains those nonmultiplicative variables which have been already used in the algorithm for construction of nonmultiplicative prolongations. This set serves to prevent the repeated prolongations.

After every insertion of a new element p in T all elements $r \in T$ such that $\mathrm{lm}(r) \succ \mathrm{lm}(p)$ are moved from T to Q in line 13. Such a displacement provides minimality of the output basis.

It should also be noted that for any triple $p \in T$ the set *vars* must always be a subset of the set of nonmultiplicative variables of $\mathrm{pol}(p)$. Line 21 controls this condition.

Algorithm JanetBasis$(F, \prec)$
Input: $F \in \mathbb{R} \setminus \{0\}$, a finite polynomial set
$\prec$, an admissible ordering
Output: G, a minimal Janet basis of $Id(F)$

```
 1: choose f ∈ F with the lowest lm(f) w.r.t. ≻
 2: T := {f, lm(f), ∅}
 3: Q := {{q, lm(q), ∅} | q ∈ F \ {f}}
 4: Q :=JanetHeadReduce(Q, T)
 5: while Q ≠ ∅ do
 6:    choose p ∈ Q such that lm(pol(p)) has no proper divisors among {lm(pol(q)) | q ∈ Q \ {p}}
 7:    if lm(pol(p)) = 1 then
 8:       return {1}
 9:    else
10:       Q := Q \ {p}
11:       if lm(pol(p)) = anc(p) then
12:          for all {r ∈ T | lm(pol(r)) ≻ lm(pol(p))} do
13:             Q := Q ∪ {r};      T := T \ {r}
14:          od
15:       fi
16:       pol(p) := NF_J(pol(p), T)
17:    fi
18:    T := T ∪ {p}
19:    for all q ∈ T and x ∈ NM_J(pol(q), T) \ nmp(q) do
20:       Q := Q ∪ {{pol(q) · x, anc(q), ∅}}
21:       nmp(q) := nmp(q) ∩ NM_J(pol(q), T) ∪ {x}
22:    od
23:    Q :=JanetHeadReduce(Q, T)
24: od
25: return G := {pol(f) | f ∈ T}
```

The initialization step is done in lines 1–4. The subalgorithm **JanetHeadReduce** performs Janet reduction of the leading terms of polynomials in Q modulo polynomials in T.

In the main loop 5–24 an element in Q is selected in line 6. The correctness of this selection strategy proved in [34]. In practice the cardinality Q at intermediate steps of the algorithm is rather large and easily runs up to hundreds and thousands. At the same time there may be different polynomials in Q with identical leading monomials. Therefore, the restriction in line 6 still admits some arbitrariness. In our implementation in [33] for the degree-reverse-lexicographical ordering a triple $p \in Q$ with the minimal $\deg(\mathrm{lm}(\mathrm{pol}(p)))$ was chosen. In the case of several such polynomials in Q, the one with the minimal number of terms was picked up.

Line 8 breaks computations in the case when inconsistency is revealed during the head term reduction in Q and returns the unit basis. In line 16 the tail Janet reduction is done, then the Janet reduced polynomial in p is inserted in T and all the higher ranked polynomials are moved to Q (loop 12-14). Actually this displacement takes place only if a polynomial in p has been subjected by the head term reduction in line 23. Otherwise, $\mathrm{pol}(p) \succ \mathrm{pol}(r)$ holds for any $r \in T$. The insertion of a new polynomial in T may generate new nonmultiplicative prolongations of elements in T which are added to Q in line 20. To avoid repeated prolongations the set $\mathrm{nmp}(q)$ of Janet nonmultiplicative variables for q has been used to construct its prolongations is enlarged with x in line 21.

The subalgorithm **JanetHeadReduce** computes the Janet head normal form of polynomials in Q modulo polynomials in T

Subalgorithm JanetHeadReduce(Q, T)
Input: Q and T, sets of triples
Output: Janet head reduced set Q modulo T

```
 1: S := Q
 2: Q := ∅
 3: while S ≠ ∅ do
 4:    choose p ∈ S
 5:    S := S \ {p}
 6:    h := HNF_J(p, T)
 7:    if h ≠ 0 then
 8:       if lm(pol(p)) ≠ lm(h) then
 9:          Q := Q ∪ {h, lm(h), ∅}
10:       else
11:          Q := Q ∪ {p}
12:       fi
13:    fi
14: od
15: return Q
```

---

and invokes in line 6 subalgorithm $\mathbf{HNF_J}(p, T)$ that does head reduction of a single polynomial p. For a head reducible input polynomial $\mathrm{pol}(f)$ the two involutive analogues of the Buchberger criteria [4] criteria are verified in line 8 of subalgorithm $\mathbf{HNF_J}$:

- **Criterion I**(f, g) is true iff $\mathrm{anc}(f) \cdot \mathrm{anc}(g) \mid \mathrm{lm}(\mathrm{pol}(f))$.
- **Criterion II**(f, g) is true iff $\deg(\mathrm{lcm}(\mathrm{anc}(f) \cdot \mathrm{anc}(g))) < \deg(\mathrm{lm}(\mathrm{pol}(f))$.

If any of the two criteria is true, then $HNF(\mathrm{pol}(f), T) = 0$ [33]. Though as shown in [35] **Criterion II** does not fully replace the Buchberger chain criterion, in practice **Criterion II** works pretty well as our computer experiments demonstrate [33].

The last subalgorithm $\mathbf{NF_J}$ performs the Janet tail reduction of a polynomial with irreducible leading term. It outputs the full Janet normal form $NF_J(f, T)$ of the input polynomial f modulo polynomial set containing in T. This subalgorithm is called in line 16 of the main algorithm **JanetBasis** and performs a chain of elementary involutive reductions until every term in the obtained polynomial becomes Janet irreducible modulo polynomials in T.

It should be noted that both the full Janet normal form and the Janet head normal form are uniquely defined and, hence, uniquely computed by the above subalgorithms. This uniqueness is a consequence of a Janet divisor among the leading terms of polynomials in T at every step of intermediate computations [29].

Subalgorithm HNF$_J(f,T)$

Input: $f = \{\mathrm{pol}(f), \mathrm{anc}(f), \mathrm{nmp}(f)\}$, a triple
T, a set of triples

Output: $h = HNF_J(\mathrm{pol}(f), T)$, the Janet head normal form of the polynomial in f modulo polynomial set in T

1: $G := \{\mathrm{pol}(g) \mid g \in T\}$
2: **if** $\mathrm{lm}(\mathrm{pol}(f))$ is involutively irreducible modulo G **then**
3: **return** f
4: **else**
5: $h := \mathrm{pol}(f)$
6: **choose** $g \in T$ such that $\mathrm{lm}(\mathrm{pol}(g)) \mid_J \mathrm{lm}(h)$
7: **if** $\mathrm{lm}(h) \neq \mathrm{anc}(f)$ **then**
8: **if CriterionI**(f,g) **or CriterionII**(f,g) **then**
9: **return** 0
10: **fi**
11: **else**
12: **while** $h \neq 0$ **and** $\mathrm{lm}(h)$ is $L-$reducible modulo G **do**
13: **choose** $q \in G$ such that $\mathrm{lm}(q) \mid_J \mathrm{lm}(h)$
14: $h := h - q \cdot \mathrm{lt}(h)/\mathrm{lt}(q)$
15: **od**
16: **fi**
17: **fi**
18: **return** h

NF$_J(f,T)$

Input: f, a polynomial such that $f := HNF_J(f,T)$;
T, a set of triples

Output: $h = NF_J(f,T)$, the full Janet normal form of h
modulo polynomial set in T

1: $G := \{\mathrm{pol}(g) \mid g \in T\}$
2: $h := f$
3: **while** $h \neq 0$ **and** h has a term t Janet reducible modulo G **do**
4: **choose** $g \in G$ such that $\mathrm{lm}(g) \mid_J t$
5: $h := h - g \cdot t/\mathrm{lt}(g)$
6: **od**
7: **return** h

4.4 Converting Bases and Finding Roots

As we emphasize in the previous section, to find common roots of polynomials in a given system it is worthwhile to compute a pure lexicographical involutive or reduced Gröbner basis. We do this computation in the following three steps:

1. Computation of a minimal degree-reverse-lexicographical Janet basis by the above described algorithm.
2. Extraction from the Janet basis obtained the reduced Gröbner basis.
3. Conversion of the degree-reverse-lexicographical Gröbner basis into the pure lexicographical one by the famous FGLM algorithm [36].

Step 2 is done immediately due to the history of prolongations stored in the polynomial triples (Sect. 4.3). Since the reduced Gröbner basis is a subset of the Janet basis computed [29] and this subset is irreducible with respect to the conventional (noninvolutive) reductions, a triple $p = \{f, u, vars\}$ in the Janet basis contains an element f of the reduced Gröbner basis if and only if $lm(f) = u$. This relation means that an element of the reduced Gröbner basis is such an element in the Janet basis that it has no ancestors in the last basis. Indeed, the leading term of this element cannot be a prolongation of the leading term of other element in the basis. In addition to the use of criteria (Section 5.2) this is one more byproduct of the triple representation.

The conversion of the degree-reverse-lexicographical Gröbner basis extracted (old basis) into the pure lexicographical basis (new basis) is done as follows [36, 37]. First, a sequence of monomials is generated, starting from the least ones w.r.t. to the new ordering and then their normal forms are computed modulo the old basis until there appears a monomial whose normal form is a linear combination of normal forms of the preceding monomials. In this case, we add the polynomial given by this relation to the new basis. This process is continued by constructing other elements in the new basis by treatment of the next variables in accordance with the new monomial order. Computation of the normal form for a monomial is simplified if one takes into account the fact that the normal forms of all its proper divisors have been computed.

A degree-reverse-lexicographical Gröbner basis admits to find roots of the initial polynomial system by the sequential solving of univariate polynomial equations. Given a univariate polynomial, we tried first to factorize it and used the built-in factorization routines of computer algebra system Reduce 3.7 [38] for this purpose. If the factorization failed to give exact roots we used a special software package ROOTs written on the top of PARI-GP system [39] to find the roots numerically for the factors obtained.

5 The Results for Polynomial Potentials with $q \leq 3$

5.1 Sextic QES Oscillator with $q = 1$ and Any N

Starting from the first nontrivial sextic-oscillator potential (7) with $q = 1$ and with the binding energies re-parametrized in accord with eq. (23) where $s_1 = s$,

$$E = \frac{1}{2}\frac{g_1}{\sqrt{g_2}}\,D + (64\,g_2)^{1/4}\,\sqrt{D}\,s\,,$$

full attention must be paid to the selfconsistency problem represented by the set of equations (25). At every N, its first nontrivial $q = 1$ version

$$\begin{pmatrix} s & 1 & & & & \\ N-1 & s & 2 & & & \\ & N-2 & s & 3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 2 & s & N-1 \\ & & & & 1 & s \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{N-2} \\ p_{N-1} \end{pmatrix} = 0 \tag{27}$$

has the form of an asymmetric eigenvalue problem. In standard manner it leads to the secular equation (20) expressible as the following sequence of the polynomial conditions,

$$s^3 - 4\,s = 0, \qquad N = 3,$$

$$s^4 - 10\,s^2 + 9 = 0, \qquad N = 4,$$

$$s^5 - 20\,s^3 + 64\,s = 0, \qquad N = 5,$$

etc. By mathematical induction, all the infinite hierarchy of these equations has been recently derived and solved in ref. [15].

Quite remarkably, all of the *real* (i.e., "physical") energy roots $s = s^{(j)}$ proved to be equal to integers. Moreover, all of them may be determined by the single and compact formula

$$s = s^{(j)} = -N - 1 + 2j, \quad j = 1, 2, \ldots, N. \tag{28}$$

One imagines that all the coefficients $p_n^{(j)}$ may be normalized to integers,

$$p_0^{(1)} = 1, \qquad N = 1,$$

$$p_0^{(1)} = p_1^{(1)} = p_0^{(2)} = -p_1^{(2)} = 1, \qquad N = 2,$$

$$p_0^{(1)} = p_2^{(1)} = p_0^{(2)} = -p_2^{(2)} = p_0^{(3)} = p_2^{(3)} = 1, \quad p_1^{(1)} = -p_1^{(3)} = 2, \quad p_1^{(2)} = 0, \quad N = 3,$$

etc.

The first result of our subsequent computations using the symbolic manipulation techniques proved equally encouraging since we succeeded in compactification of the set of the above recurrent solutions to the single leading-order form of the related wave functions,

$$\psi^{(j)}(r) = r^{\ell+1} \left(1 + \frac{r^2}{\mu}\right)^{N-j} \left(1 - \frac{r^2}{\mu}\right)^{j-1} \exp\left(-\frac{1}{2}\alpha_0 r^2 - \frac{1}{4}\alpha_1 r^4\right),$$

$$j = 1, 2, \ldots, N. \tag{29}$$

A few more comments may be added. Firstly, the large and degenerate nodal zeros in eq. (29) are a mere artifact of the zero-order construction. This means that the apparently interesting exact summability of all the separate $\mathcal{O}(r^2/\mu)$ error terms is not too relevant, indeed. Although it leads to the zero-order nodes at $r = \mathcal{O}(\sqrt{\mu}) = \mathcal{O}(D^{1/4})$, these nodes have no real physical meaning.

Secondly, the leading-order perturbative approximation provides a reliable information about the energies. They are asymptotically degenerate, due to the large overall shift of the energy scale as explained in section 3.2. In addition, the next-order corrections may be easily obtained by the recipes of the textbook perturbation theory. As long as the coefficients p_n are defined in integer arithmetics, the latter strategy gives, by construction, all the above-mentioned energy corrections without any rounding errors in a way outlined in more detail in ref. [15].

In the other words, we may say that formula (29) may either be truncated to its leading-order form $\psi^{(j)}(r) = r^{\ell+1} \exp\left(-\lambda^{(2)}(r)\right)$ or, better, its full form may be used as a generating function which facilitates the explicit evaluation of the coefficients $p_n^{(j)}$. In comparison, both the oversimplified harmonic oscillator and the $q = 1$ wave functions may be characterized by the similar coordinate dependence which becomes spurious (i.e., dependent on the selected normalization) everywhere beyond the perturbatively accessible domain of r.

The energies specified by eq. (28) form an amazingly regular multiplet. A natural question arises whether a similar regularity could re-emerge at the larger integer indices $q > 1$. We are now going to demonstrate that in spite of the growth of the technical obstacles in dealing with the corresponding key equation (25), the answer is, definitely, affirmative.

5.2 Decadic Oscillators with $q = 2$ and Any N

The decadic anharmonic oscillator exhibits certain solvability features which motivated its deeper study in non-Hermitian context [40]. The changes of variables make this oscillator very closely related to the common quartic problem [14, 6]. Paying attention to the $D \gg 1$ domain and abbreviating the parameters $s_1 = s$ and $s_2 = t$ of the respective decadic-oscillator energy and coupling in eq. (23), we arrive at the four-diagonal version of our solvability condition (25) at $q = 2$,

$$\begin{pmatrix}
s & 1 & & & & & \\
t & s & 2 & & & & \\
N-1 & t & s & 3 & & & \\
 & N-2 & t & s & 4 & & \\
 & & \ddots & \ddots & \ddots & \ddots & \\
 & & & 3 & t & s & N-1 \\
 & & & & 2 & t & s \\
 & & & & & 1 & t
\end{pmatrix}
\begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{N-2} \\ p_{N-1} \end{pmatrix} = 0 \,. \tag{30}$$

This is the first really nontrivial equation of the class (25). In order to understand its algebraic structure in more detail, let us first choose the trivial case with $N = 2$ and imagine that the resulting problem (with $p_1 \neq 0$ due to the definition of N) may be solved by the determination of the unknown ratio of the wave-function coefficients $p_0/p_1 = -t$ from the last line, and by the subsequent elimination of $t = 1/s$ using the first line. The insertion of these two quantities transforms the remaining middle line into the cubic algebraic equation $s^3 = 1$ with the single real root $s = 1$.

The next equation at $N = 3$ is still worth mentioning because it shows that the strategy accepted in the previous step is not optimal. Indeed, the same elimination of $p_1/p_2 = -t$ and of $p_0/p_2 = (t^2 - s)/2$ from the third line leads to the apparently ugly result

$$\begin{aligned} st^2 - s^2 - 2t &= 0, \\ t^3 - 3st + 4 &= 0. \end{aligned}$$

An alternative strategy starting from the elimination of p_0 and p_2 leads to the much more symmetric pair of the conditions

$$\begin{aligned} t^2 - s^2 t + 2s &= 0, \\ s^2 - t^2 s + 2t &= 0 \end{aligned}$$

the respective pre-multiplication of which by t and s gives the difference $t^3 = s^3$. This means that $t = \varepsilon\, s$ where the three eligible proportionality constants exist such that $\varepsilon^3 = 1$. Thus, our problem degenerates to a quadratic equation with the pair of the real roots $s = t = s^{(1,2)}$ such that

$$s^{(1)} = 2, \qquad s^{(2)} = -1. \tag{31}$$

The "ugliness" of the procedure of elimination is inessential as long as we can produce the results by any "brute-force" symbolic manipulations on the computer [8].

Once we encountered the limitations of the naive algorithms, we were forced to pay attention to all the above-described sophistications of our algorithms. Fortunately, this overall strategy proved successful. Using the methods described in preceding sections we revealed that the step-by-step elimination of the redundant unknowns gives the best form of the results when one uses the Janet bases.

One of the main and most important byproducts of our approach is that the resulting final effective or "secular" polynomial equations for the single unknown quantity s depend in practice on its power s^{q+1} only. The clear illustration is provided by the present $q = 2$ case at $N = 3$ giving the rule

$$s^6 - 7\, s^3 - 8 = 0, \qquad N = 3. \tag{32}$$

This equation possesses the same complete set of the real roots (31) of course. Still, what is important is that *any* root $r = s_0^3$ of eq. (32) itself still represents just the third (and in the more general cases, $(q + 1)$st) power of the final relevant quantity with the physical meaning of the

energy. Thus, we still have to solve the relation $r = s_0^3$ where merely the value of r is known and where, therefore, *at most one* final parameter s_0 is real (= acceptable).

One may conclude that the real energies of the "strongly spiked" decadic oscillator are very easily determined even without a detailed specification of an "optimal" elimination pattern, and that it is very easy to get rid of the redundant non-real roots s_0 at the very end of the algorithm. Of course, the numerous complex roots should not be discarded *a priori* as they might prove important in some other applications like a systematic computation of the corrections [15] which were not mentioned in our present paper at all.

Our conclusions extracted at $N = 3$ are confirmed at the next N leading to the effective polynomial equation

$$s^{10} - 27\,s^7 + 27\,s^4 - 729\,s = 0\,, \quad N = 4\,.$$

Being tractable by our newly developed computer software and playing still the role of a test, it results in the set of the mere two real roots again,

$$s^{(1)} = 3, \qquad s^{(2)} = 0, \qquad N = 4, \quad q = 2.$$

One finds that the $q = 2$ problem may be reduced to a single polynomial equation with $\begin{pmatrix} N+1 \\ 2 \end{pmatrix}$ complex roots s at any N. The explicit calculations may be summarized in a statement that *all* the general physical (i.e., real) spectrum of energies proves to be quite rich and appears described by the closed and amazingly simple and transparent formula again,

$$s^{(j)} = N + 2 - 3j, \qquad j = 1, 2, \ldots, j_{max}, \qquad j_{max} = entier\left[\frac{N+1}{2}\right]\,. \tag{33}$$

After one applies our Janet-basis procedure at the higher and higher dimensions N, one repeatedly arrives at the confirmation of the $N-$independent empirical observation (33) and extends it by another rule that at all the values of the dimension N, there exist only such real roots that $s^{(j)} = t^{(j)}$. This means that each "solvability admitting" real energy s requires, purely constructively, the choice of its own "solvability admitting" real coupling constant t.

5.3 Oscillators with $q = 3$ and Their Solution at Any N

At $q = 3$ we have to solve the five-diagonal eq. (25),

$$\begin{pmatrix}
r & 1 & & & & & & \\
s & r & 2 & & & & & \\
t & s & r & 3 & & & & \\
N-1 & t & s & r & 4 & & & \\
 & N-2 & t & s & r & 5 & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & 4 & t & s & r & N-1 \\
 & & & & 3 & t & s & r \\
 & & & & & 2 & t & s \\
 & & & & & & 1 & t
\end{pmatrix}
\begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{N-2} \\ p_{N-1} \end{pmatrix} = 0 \tag{34}$$

which may be reduced, by means of the similar symbolic computations as above, to the single polynomial problem

$$t^9 - 12\,t^5 - 64\,t = 0$$

at $N = 3$, to the next similar condition

$$t^{16} - 68\,t^{12} - 442\,t^8 - 50116\,t^4 + 50625 = 0$$

at $N = 4$, to the conditions of vanishing of the secular polynomial

$$t^{25} - 260\,t^{21} + 7280\,t^{17} - 1039040\,t^{13} - 152089600\,t^9 + 2030239744\,t^5 + 10485760000\,t$$

at $N = 5$, or to the perceivably longer equation

$$t^{36} - 777\,t^{32} + 135716\,t^{28} - 17189460\,t^{24} - 3513570690\,t^{20} -$$

$$-1198527160446\,t^{16} + 103857100871252\,t^{12} + 873415814269404\,t^{8} +$$

$$+74500845455535625\,t^{4} - 75476916312890625 = 0$$

at $N = 6$ etc. These computations represent a difficult technical task but at the end they reveal again a clear pattern in the structure of the secular polynomials as well as in their solutions. One arrives at the similar final closed formulae as above. Now one only deals with more variables so that we need two indices to prescribe the complete classification scheme

$$\begin{aligned} s = s^{(j)} &= N + 3 - 4j, \\ r = r^{(j,k)} = t = t^{(j,k)} &= -N - 3 + 2j + 2k, \\ k = 1, 2, \ldots, k_{max}(j), &\qquad k_{max}(j) = N + 2 - 2j\ , \\ j = 1, 2, \ldots, j_{max}\ , &\qquad j_{max} = entier\left[\frac{N+1}{2}\right]\ . \end{aligned} \tag{35}$$

We may re-emphasize that all the real roots share the symmetry $r = t$ but admit now a different second root s. The physical meaning of these roots is obvious. Thus, the energies of the oscillations in the polynomial well

$$V^{(q=3,k=1)}(r) = a\,r^2 + b\,r^4 + \ldots + g\,r^{14}$$

will be proportional to the roots $r^{(j,k)}$. After the change of variables, the roots $s^{(j)}$ will represent energies for the alternative, "charged" polynomial potentials

$$V^{(q=3,k=2)}(r) = \frac{e}{r} + a\,r + b\,r^2 + \ldots + f\,r^6$$

etc [22].

6 The Results with $q = 4$ and $q = 5$ for $N \leq N_{max}$

6.1 Non-Integer Roots Emerging at $q = 4$ and $N \leq 6$

In our present formulation of the problem (25), we denote the descending diagonals as s_m with $m = 1, 2, 3, 4$ and get the equation

$$\begin{pmatrix} s_1 & 1 & & & \\ s_2 & \ddots & \ddots & & \\ s_3 & \ddots & \ddots & N-1 & \\ s_4 & \ddots & \ddots & s_1 & \\ N-1 & \ddots & \ddots & s_2 & \\ & \ddots & \ddots & s_3 & \\ & & 1 & s_4 & \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{N-1} \end{pmatrix} = 0\ . \tag{36}$$

Its systematic solution does not parallel completely the above-described procedures. In fact, the reduction of the problem to the search for the roots of a single polynomial secular equation $P(x) = 0$ (in the selected auxiliary variable $x = -s_4$) enables us only to factorize $P(x)$ on an extension of the domain of integers,

$$P(x) = (x+3)\left(2\,x + 1 - \sqrt{5}\right)\left(2\,x + 1 + \sqrt{5}\right)$$

$$\left(2\,x^2 - 3\,x + 3\,\sqrt{5}x + 18\right)\left(2\,x^2 - 3\,x - 3\,\sqrt{5}x + 18\right)$$

$$\left(2\,x^2-3\,x-\sqrt{5}x+8+2\,\sqrt{5}\right)\left(2\,x^2-3\,x+\sqrt{5}x+8-2\,\sqrt{5}\right)$$
$$\left(x^2+x+\sqrt{5}x+4+\sqrt{5}\right)\left(x^2+x-\sqrt{5}x+4-\sqrt{5}\right)$$
$$\left(-2\,\sqrt{5}+8-3\,x+3\,\sqrt{5}x+2\,x^2\right)\left(2\,\sqrt{5}+8-3\,x-3\,\sqrt{5}x+2\,x^2\right)$$
$$\left(-2\,\sqrt{5}+8+7\,x-\sqrt{5}x+2\,x^2\right)\left(2\,\sqrt{5}+8+7\,x+\sqrt{5}x+2\,x^2\right)$$
$$\left(2\,x^2+2\,x+3-\sqrt{5}\right)\left(2\,x^2+2\,x+3+\sqrt{5}\right)$$
$$\left(\sqrt{5}+3-3\,x-\sqrt{5}x+2\,x^2\right)\left(-\sqrt{5}+3-3\,x+\sqrt{5}x+2\,x^2\right)$$
$$\left(2\,\sqrt{5}+8-3\,x+\sqrt{5}x+2\,x^2\right)\left(-2\,\sqrt{5}+8-3\,x-\sqrt{5}x+2\,x^2\right)\ .$$

From this lengthy formula it follows that we get

$$s_4^{(1)}=3,\qquad s_4^{(2)}=\frac{\sqrt{5}+1}{2}\approx 1.618,\qquad s_4^{(3)}=\frac{\sqrt{5}-1}{2}\approx -0.618\ .$$

There only exist these three real roots s_4 in this case.

The similar computerized procedure gave us the real roots also at $N=5$ and $N=6$. The details may be found in ref. [17]. The inspection of these results leads to the conclusion that $s_2=s_3$ and $s_1=s_4$. We did not succeed in an application of our algorithms beyond $N=6$ yet. The reason is that even the $N=5$ version of eq. (36) in its reduction to the condition

$$x^{70}-936\,x^{65}+67116\,x^{60}-95924361\,x^{55}-74979131949\,x^{50}+8568894879002\,x^{45}-$$
$$\ldots-1745947227450187022233 6\,x^5+142630535951654322176=0$$

of the vanishing auxiliary polynomial required a fairly long computation for its (still closed and compact) symbolic-manipulation factorization summarized in Table 1 of ref. [17].

6.2 A Mind-Boggling Return of Integer Roots at $q=5$

$\boldsymbol{N=6}$ At $q=5$ and $N=6$ the symbolic manipulations using the Gröbner bases [4] generate the secular polynomial in $x=s_5$ which has the slightly deterring form

$$x^{91}-16120\,x^{85}+49490694\,x^{79}-286066906320\,x^{73}-3553475147614293\,x^{67}-$$
$$\ldots-319213100611990814833843025405983064064000000\,x=0\ .$$

Fortunately, it proves proportional to the polynomial with the mere equidistant and simple real zeros,

$$P_1^{(6)}(x)=x\left(x^2-1\right)\left(x^2-2^2\right)\left(x^2-3^2\right)\left(x^2-4^2\right)\left(x^2-5^2\right)\ .$$

The rest of the secular polynomial is a product of the other two elementary and positive definite polynomial factors

$$P_2^{(6)}(x)=\prod_{k=1}^{2}\left(x^2-3k\,x+3k^2\right)\left(x^2+3k^2\right)\left(x^2+3k\,x+3k^2\right)$$

and

$$P_3^{(6)}=\prod_{k=1}^{5}\left(x^2-k\,x+k^2\right)\left(x^2+k\,x+k^2\right),$$

with another positive definite polynomial

$$P_4^{(6)}=\prod_{k=1}^{12}\left(x^2-b_k\,x+c_k\right)\left(x^2+b_k\,x+c_k\right)$$

where the structure of the two series of coefficients (see their list in ref. [17]) is entirely enigmatic.

The subsequent symbolic manipulations reveal a symmetry $s_2=s_4$ and $s_1=s_5$ of all the real eigenvalues. In the $N=6$ pattern summarized in ref. [17] we recognize a clear indication of a tendency of a return to the transparency of the $q\leq 3$ results which may be written and manipulated in integer arithmetics. For obtaining a deeper insight we must move to the higher N.

$N = 7$ One should note that in spite of its utterly transparent form, the latter result required a fairly long computing time for its derivation. One encounters new technical challenges here. Indeed, the comparison of the $N = 6$ secular polynomial equation with its immediate $N = 7$ descendant

$$x^{127} - 60071\,x^{121} + 1021190617\,x^{115} - 11387407144495\,x^{109} - \ldots + c\,x \cdot 10^6 = 0$$

shows that the last coefficient

$$c = 125371220122726667620073789326658415654595883041274311330630729728$$

fills now almost the whole line. This case failed to be tractable by our current computer code and offers the best illustration of the quick growth of the complexity of the $q \geq 5$ constructions with the growth of the QES dimension parameter N.

Fortunately, we are still able to keep the trace of the pattern revealed at $N = 6$. Indeed, our new secular $N = 7$ polynomial factorizes again in the product of the four factors $P_j(x)$, $j = 1, 2, 3, 4$ where only the first one has the real zeros,

$$P_1^{(7)}(x) = P_1^{(6)}(x) \cdot \left(x^2 - 6^2\right) \ .$$

The further three factors fit the structure of their respective predecessors very well,

$$P_2^{(7)}(x) = P_2^{(6)}(x) \cdot \left(x^2 - 9\,x + 27\right)\left(x^2 + 27\right)\left(x^2 + 9\,x + 27\right)$$

and

$$P_3^{(7)} = P_3^{(6)} \cdot \left(x^2 - 6\,x + 36\right)\left(x^2 + 6\,x + 36\right)$$

while

$$P_4^{(7)} = P_4^{(6)} \cdot \prod_{k=1}^{6} \left(x^2 - f_k\,x + g_k\right)\left(x^2 + f_k\,x + g_k\right) \ .$$

The subscript-dependence of the new coefficients may be found in ref. [17] again. The key importance of the explicit knowledge of these coefficients lies in the possibility of a rigorous proof that the related roots are all complex and, hence, irrelevant from our present point of view.

$N = 8$ and $N = 9$ The growth of the degree of our secular univariate polynomials makes it quite difficult to move too far with N. One may be more explicit in this respect: In place of the numerous irregularities encountered at $q = 4$, we may now be surprised by the re-emergence of the following *closed* and very transparent elementary formula for the $q = 5$ "energies",

$$s_5 \in (-N+1, -N+2, \ldots, N-2, N-1)\,. \tag{37}$$

which is valid again for *any* integer N in a way which parallels and complements the above-mentioned results which were available and published in our previous papers [15], [6] and [17] for the Magyari's $D \gg 1$ potentials with $q = 1$, $q = 2$ and $q = 3$, respectively. In this context, their extension (37) is a brand new result which has not been published yet. Its unexplained equidistance property may be added to the list of the unresolved puzzles related to the Magyari equations. Indeed, the equidistance exemplified by eq. (37) would reflect a hidden algebra in linear cases but what is most intriguing here is the fact that the present Magyari equations are *non-linear*!

Another challenging feature of the problem lies in its exact solvability which is based on the factorization of polynomials of a very large degree $\mathcal{D}$ which grows, moreover, very quickly with N. Empirically, this degree may be even specified by the closed formula at $q = 5$ where $\mathcal{D} = 3N^2 - 3N + 1$ in a way illustrated by the next two explicit secular equations

$$s_5^{169} - 186238\,s_5^{163} + 11768813199\,s_5^{157} - \ldots = 0, \qquad N = 8, \tag{38}$$

$$s_5^{217} - 502386\,s_5^{211} + 94933635261\,s_5^{205} - \ldots = 0, \qquad N = 9. \tag{39}$$

On the basis of these observations we may conclude that an overall pattern of the smooth $N-$dependence of the equations survives, *mutatis mutandis*, smoothly the transition to the higher N.

One can also prove (at least up to $N \leq 9$ at present) by construction that *all* the other factors of the secular polynomial have an elementary quadratic-polynomial form and remain positive for all the real "re-scaled energies" s_5. Their coefficients are elementary (we skip the examples here) so that the strict proof that they possess no real zeros is also elementary and very quick (one just evaluates the discriminants). A full parallelism between all $N \leq 9$ is achieved and we might conjecture, on this background, the possibility of its extension to all the non-negative integers N. A strict proof of this conjecture could probably be based on mathematical induction but we do not feel it really urgent at the moment.

7 Summary

It is rather amusing to imagine that the *majority* of quantitative predictions in nuclear, atomic, molecular and condensed matter physics must rely on a more or less purely numerical model. The completely non-numerically tractable quantum systems are rare though, at the same time, useful and transparent (cf., e.g., the above-mentioned description of vibrations in molecules mimicked by harmonic oscillators). In our present paper we revealed that in the domain of the large spatial dimension $D \gg 1$, the class of the exactly solvable models becomes, in a certain sense, broader. Thus, one might call *all* the polynomially anharmonic oscillators "asymptotically solvable".

This is an important and also not yet fully appreciated observation obtained due to the lasting advancement of the computer algebra and related software as described in more detail in Sections 4 and 5. A fairly universal apparatus of these sections was reported in close connection with its application to our Magyari-type equations (25).

In a certain perspective we found new closed solutions of Schrödinger equation with polynomial potentials in the domain of the large angular momenta $\ell \gg 1$ where alternative techniques are also available (cf., e.g., their review [41] and/or very recent discussion [42]). Our results revealed the existence and provided the construction of certain fairly large multiplets of "exceptional" $\ell \gg 1$ bound states for a very broad class of polynomial oscillators. We believe that they might find an immediate application in some phenomenological $D \gg 1$ models.

From the mathematical point of view, the most innovative and characteristic feature of our new $D \gg 1$ QES multiplets lies in the existence of the new *closed and compact* formulae for the QES energies and/or couplings *at all* N. For this reason, the corresponding partially solvable polynomial oscillator Hamiltonians $H_0^{(q,N)}$ might even be understood as lying in the QES class as its new and fairly specific subclass.

Due to an exceptional transparency of our constructions of $H_0^{(q,N)}$, a facilitated return to the "more realistic" finite spatial dimensions $D = \mathcal{O}(1)$ might prove tractable by perturbation techniques. Two reasons may be given in favor of such a strategy. First, due to the specific character of our present "unperturbed" spectra *and* eigenvectors, the perturbation algorithm might be implemented *in integer arithmetics* (i.e., without rounding errors) in a way outlined, preliminarily, in ref. [15] at $q = 1$. Second, the evaluation of the few lowest orders might suffice. This expectation follows from the enhanced flexibility of the available zero-order Hamiltonians. *A priori*, a better convergence of the corrections might be expected to result from a better quality of a "guaranteed smallness" of the difference between a given Hamiltonian H at a finite D and one of its present $D = \infty$ QES approximants H_0.

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