AN ALGEBRAIC CHARACTERIZATION OF UNIQUELY VERTEX COLORABLE GRAPHS

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Abstract. The study of graph vertex colorability from an algebraic perspective has introduced novel techniques and algorithms into the field. For instance, k-colorability of a graph can be characterized in terms of whether its graph polynomial is contained in a certain ideal. In this paper, we interpret unique colorability in an analogous manner and prove an algebraic characterization for uniquely k-colorable graphs. Our result also gives algorithms for testing unique colorability. As an application, we verify a counterexample to a conjecture of Xu concerning uniquely 3-colorable graphs without triangles.

1. Introduction

Let G be a simple, undirected graph with vertices $V = \{1, \ldots, n\}$ and edges E. The *graph polynomial* of G is given by

$$f_G = \prod_{\substack{\{i,j\} \in E, \\ i<j}} (x_i - x_j).$$

Fix a positive integer $k < n$, and let $C_k = \{c_1, \ldots, c_k\}$ be a k-element set. Each element of C_k is called a *color*. A (vertex) *k-coloring* of G is a map $\nu : V \to C_k$. We say that a k-coloring ν is *proper* if adjacent vertices receive different colors; otherwise ν is called *improper*. The graph G is said to be *k-colorable* if there exists a proper k-coloring of G.

Let $\Bbbk$ be an algebraically closed field of characteristic not dividing k that contains the kth roots of unity. Also, set $R = \Bbbk[x_1, \ldots, x_n]$ to be the polynomial ring over $\Bbbk$ in indeterminates $x_1, \ldots, x_n$. Let $\mathcal{H}$ be the set of graphs with vertices $\{1, \ldots, n\}$ consisting of a clique of size $k+1$ and isolated vertices. We will be interested in the following ideals of R:

$$\begin{aligned}
J_{n,k} &= \langle f_H : H \in \mathcal{H} \rangle, \\
I_{n,k} &= \langle x_i^k - 1 : i \in V \rangle, \\
I_{G,k} &= I_{n,k} + \langle x_i^{k-1} + x_i^{k-2}x_j + \cdots + x_i x_j^{k-2} + x_j^{k-1} : \{i,j\} \in E \rangle.
\end{aligned}$$

One should think of $I_{n,k}$ and $I_{G,k}$ as representing the set of all k-colorings and proper k-colorings of the graph G, respectively. These ideals are important because

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they allow for an algebraic formulation of k-colorability. The following theorem collects the results in the series of papers [2, 3, 10, 11, 12].

Theorem 1.1. *The following four statements are equivalent:*

(1) *The graph G is not k-colorable.*
(2) *The constant polynomial* 1 *belongs to the ideal $I_{G,k}$.*
(3) *The graph polynomial f_G belongs to the ideal $I_{n,k}$.*
(4) *The graph polynomial f_G belongs to the ideal $J_{n,k}$.*

The next result says that the generators for the ideal $J_{G,k}$ in the above theorem are very special (see Section 2 for a review of the relevant definitions). A proof can be found in [10].

Theorem 1.2 (J. de Loera)**.** *The set of polynomials, $\{f_H : H \in \mathcal{H}\}$, is a universal Gröbner basis of $J_{n,k}$.*

Remark 1.3. The set $\mathcal{G} = \{x_1^k - 1, \ldots, x_n^k - 1\}$ is a universal Gröbner basis of $I_{n,k}$, but this follows easily since the leading terms of $\mathcal{G}$ are relatively prime (find reference), regardless of term order.

We give a self-contained proof of Theorem 1.1 in Section 2. We say that a graph is *uniquely k-colorable* if there is a unique proper k-coloring up to permutation of the colors in C_k. In this case, partitions of the vertices into subsets having the same color are the same for each of the $k!$ proper colorings of G. A natural refinement of Theorem 1.1 would be an algebraic characterization of when a k-colorable graph is uniquely k-colorable. Our main result provides such a characterization.

Theorem 1.4. *Let G be k-colorable with a coloring that uses all k colors. Then there exist polynomials $g_1, \ldots, g_n$, and g such that the following three statements are equivalent:*

(1) *The graph G is uniquely k-colorable.*
(2) *The polynomials $g_1, \ldots, g_n$ belong to the ideal $I_{G,k}$.*
(3) *The graph polynomial f_G belongs to the ideal $I_{n,k} + \langle g \rangle$.*

Remark 1.5. The polynomials $g_1, \ldots, g_n$, and g can be written down explicitly once we are given a proper k-coloring of G; we will define them in Section 4. See also Example 1.7 below.

For a uniquely colorable graph, the polynomials $g_1, \ldots, g_n$ in the theorem are especially nice, yielding a partial analog to Theorem 1.2.

Theorem 1.6. *Let G be uniquely k-colorable. Then the polynomials $g_1, \ldots, g_n$ form the reduced Gröbner basis for $I_{G,k}$ with respect to the lexicographic order with $x_n \prec \cdots \prec x_1$.*

Example 1.7. We present an example of a uniquely 3-colorable graph on $n = 12$ vertices and give the polynomials $g_1, \ldots, g_n$ from Theorem 1.4.

Let G be the graph given in Figure 1. The following set of 12 polynomials is a reduced Gröbner basis for the ideal $I_{G,k}$. The leading terms of each g_i are underlined.

$$\begin{aligned}
&\{\underline{x_{12}^3} - 1,\ \underline{x_7} - x_{12},\ \underline{x_4} - x_{12},\ \underline{x_3} - x_{12},\\
&\ \underline{x_{11}^2} + x_{11}x_{12} + x_{12}^2,\ \underline{x_9} - x_{11},\ \underline{x_6} - x_{11},\ \underline{x_2} - x_{11},\\
&\ \underline{x_{10}} + x_{11} + x_{12},\ \underline{x_8} + x_{11} + x_{12},\ \underline{x_5} + x_{11} + x_{12},\ \underline{x_1} + x_{11} + x_{12}\}.
\end{aligned}$$

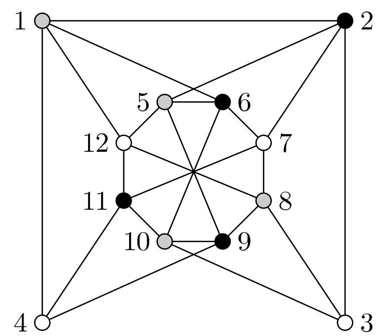

FIGURE 1. A uniquely 3-colorable graph without triangles

Notice that the leading terms of the polynomials in each line above correspond to the different color classes of the coloring of G. $\square$

The organization of this paper is as follows. In Section 2, we discuss some of the algebraic tools that will go into the proofs of our main results. Section 3 is devoted to a proof of Theorem 1.1, and in Section 4, we present arguments for Theorems 1.4 and 1.6. Theorems 1.1 and 1.4 give algorithms for testing k-colorability and unique k-colorability of graphs, and we discuss the implementation of them in Section 5, along with a verification of a counterexample [1] to a conjecture [4, 7, 13] by Xu.

2. Algebraic Preliminaries

We briefly review the basic concepts of commutative algebra that will be useful for us here. Let I be an ideal of $R = \Bbbk[x_1, \ldots, x_n]$. The *variety* $V(I)$ of I is the set of points in $\Bbbk^n$ that are zeroes of all the polynomials in I. Conversely, the *vanishing ideal* $I(V)$ of a set $V \subseteq \Bbbk^n$ is the ideal of those polynomials vanishing on all of V. These two definitions are related by way of $V(I(V)) = V$ and $I(V(I)) = \sqrt{I}$, in which

$$\sqrt{I} = \{f : f^n \in I \text{ for some } n\}$$

is the *radical* of I. The ideal I is called *zero-dimensional* if $V(I)$ is finite. A *term order* $\prec$ for the monomials of R is a well-ordering which is multiplicative and for which the constant monomial is smallest. The *initial term* (or *leading monomial*) $in_{\prec}(f)$ of a polynomial $f \in R$ is the largest monomial in f with respect to $\prec$. The *standard monomials* $\mathcal{B}_{\prec}(I)$ of I are those monomials which are not the initial terms of any polynomial in I.

Many arguments in commutative algebra and algebraic geometry are simplified when restricted to radical, zero-dimensional ideals (resp. multiplicity-free, finite varieties), and those found in this paper are not exceptions. The following fact is useful in this regard.

Lemma 2.1. *Let I be a zero-dimensional ideal and fix a term order $\prec$. Then $\dim_{\Bbbk} R/I = |\mathcal{B}_{\prec}(I)| \geq |V(I)|$. Furthermore, the following are equivalent:*

(1) *I is a radical ideal.*
(2) *I contains a univariate square-free polynomial in each indeterminate.*
(3) *$|\mathcal{B}_{\prec}(I)| = |V(I)|$.*

Proof. See [5]. □

A finite subset $\mathcal{G}$ of an ideal I is a *Gröbner basis* (with respect to $\prec$) if the *initial ideal*,

$$in_{\prec}(I) = \langle in_{\prec}(f) : f \in I \rangle,$$

is generated by the initial terms of elements of $\mathcal{G}$. Furthermore, a *universal Gröbner basis* is a set polynomials which is a Gröbner basis with respect to all term orders. Many of the properties of I and $V(I)$ can be calculated by finding a Gröbner basis for I, and such generating sets are fundamental for computation (including the algorithms presented in the last section).

Finally, a useful operation on two ideals I and J is the construction of the *colon ideal* $I : J = \{h \in R : hJ \subseteq I\}$. If V and W are two varieties, then the colon ideal

$$I(V) : I(W) = I(V \backslash W) \tag{2.1}$$

corresponds to a set difference.

3. Vertex Colorability

In what follows, the set of colors C_k will be the set of kth roots of unity, and we will freely speak of points in $\Bbbk^n$ with all coordinates in C_k as colorings of G. In this case, a point $(v_1, \ldots, v_n) \in \Bbbk^n$ corresponds to a coloring of vertex i with color v_i for $i = 1, \ldots, n$. The varieties corresponding to the ideals $I_{n,k}$, $I_{G,k}$, and $I_{n,k} + \langle f_G \rangle$ partition the k-colorings of G as follows.

Lemma 3.1. *The varieties $V(I_{n,k})$, $V(I_{G,k})$, and $V(I_{n,k} + \langle f_G \rangle)$ are in bijection with all, the proper, and the improper k-colorings of G, respectively.*

Proof. The points in $V(I_{n,k})$ are all n-tuples of kth roots of unity and therefore naturally correspond to all k-colorings of G. Let $\mathbf{v} = (v_1, \ldots, v_n) \in V(I_{G,k})$; we must show that it corresponds to a proper coloring of G. Let $\{i, j\} \in E$ and set

$$q_{ij} = \frac{x_i^k - x_j^k}{x_i - x_j} \in I_{G,k}.$$

If $v_i = v_j$, then $q_{ij}(\mathbf{v}) = kv_i^{k-1} \neq 0$. Thus, the coloring $\mathbf{v}$ is proper. Conversely, suppose that $\mathbf{v} = (v_1, \ldots, v_n)$ is a proper coloring of G. Then, since

$$q_{ij}(\mathbf{v})(v_i - v_j) = (v_i^k - v_j^k) = 1 - 1 = 0,$$

it follows that for $\{i, j\} \in E$, we have $q_{ij}(\mathbf{v}) = 0$. This shows that $\mathbf{v} \in V(I_{G,k})$. Finally, if $\mathbf{v}$ is an improper coloring, then it is easy to see that $f_G(\mathbf{v}) = 0$, and that moreover, any $\mathbf{v} \in V(I_{n,k})$ for which $f_G(\mathbf{v}) = 0$ has two coordinates, corresponding to an edge in G, that are equal. □

The next result follows directly from Lemma 2.1. It will prove useful in simplifying many of the proofs in this paper.

Lemma 3.2. *The ideals $I_{n,k}$, $I_{G,k}$, and $I_{n,k} + \langle f_G \rangle$ are radical.*

We next describe a relationship between $I_{n,k}$, $I_{G,k}$, and $I_{n,k} + \langle f_G \rangle$.

Lemma 3.3. $I_{n,k} : I_{G,k} = I_{n,k} + \langle f_G \rangle$.

Proof. Let V and W be the set of all colorings and proper colorings, respectively, of the graph G. Now apply Lemma 3.1 and Lemma 3.2 to equation (2.1). □

The dimensions of the residue rings corresponding to these ideals are readily computed from the above discussion. Recall that the *chromatic polynomial* χ_G is the univariate polynomial for which $\chi_G(k)$ is the number of proper k-colorings of G.

Lemma 3.4. *Let χ_G be the chromatic polynomial of G. Then*

$$\chi_G(k) = \dim_{\Bbbk} R/I_{G,k},$$
$$k^n - \chi_G(k) = \dim_{\Bbbk} R/(I_{n,k} + \langle f_G \rangle).$$

Proof. Both equalities follow from Lemmas 2.1 and 3.1. □

Let $K_{n,k}$ be the ideal of all polynomials $f \in R$ such that $f(v_1, \ldots, v_n) = 0$, $(v_1, \ldots, v_n) \in \Bbbk^n$, if at most k of the v_i are distinct. Clearly, $J_{n,k} \subseteq K_{n,k}$. We will need the following result of Kleitman and Lovasz [11].

Theorem 3.5. *The ideals $K_{n,k}$ and $J_{n,k}$ are the same.*

Proof. We sketch the proof (see [11] for more details). Let $f \in K_{n,k}$ and for each subset $S \subseteq \{1, \ldots, n-1\}$, let f_S be the polynomial gotten from substituting x_n for each x_i with $i \in S$. Since $f_S \in K_{n,k}$, induction on the number of indeterminates n implies that $f_S \in J_{n,k}$ for nonempty S. If $p = x_1^{a_1} \cdots x_n^{a_n}$ is a monomial, then

$$q(p) := \sum_S (-1)^{|S|} p_S, \quad p \in R,$$

equals $(x_1^{a_1} - x_n^{a_1}) \cdots (x_{n-1}^{a_{n-1}} - x_n^{a_{n-1}}) x_n^{a_n}$. By linearity, therefore, it follows that $q = q(f) = (x_1 - x_n) \cdots (x_{n-1} - x_n) h$ for some $h \in R$. Since $q \in K_{n,k}$, the polynomial h is zero whenever at most $k-1$ of the $x_1, \ldots, x_{n-1}$ are distinct. Thus, upon expanding h in terms of powers of x_n, the coefficients will belong to $K_{n-1,k-1}$, and by induction, we may assume they all belong to $J_{n-1,k-1}$. Hence, $q \in J_{n,k}$. Finally, we have $f = q - \sum_{S \neq \emptyset} (-1)^{|S|} f_S \in J_{n,k}$, completing the proof. □

We now present a proof of Theorem 1.1.

Proof of Theorem 1.1. (1) ⇒ (2): Suppose that G is not k-colorable. Then it follows from Lemma 3.4 that $\dim_{\Bbbk} R/I_{G,k} = 0$ and so $1 \in I_G$. (2) ⇒ (3): Suppose that $I_{G,k} = \langle 1 \rangle$ so that $I_{n,k} : I_{G,k} = I_{n,k}$. Then Lemma 3.3 implies that $I_{n,k} + \langle f_G \rangle = I_{n,k}$ and hence $f_G \in I_{n,k}$. (3) ⇒ (1): Assume that f_G belongs to the ideal $I_{n,k}$. Then $I_{n,k} + \langle f_G \rangle = I_{n,k}$, and it follows from Lemma 3.4 that $k^n - \chi_G(k) = k^n$. Therefore, $\chi_G(k) = 0$ as desired. (4) ⇒ (1): Suppose that $f_G \in J_{n,k}$. Then from Theorem 3.5, there can be no proper coloring $\mathbf{v}$ (there are at most k distinct coordinates). (1) ⇒ (4): If G is not k-colorable, then for every substitution $\mathbf{v} \in \Bbbk^n$ with at most k distinct coordinates, we must have $f_G(\mathbf{v}) = 0$. It follows that $f_G \in J_{n,k}$ from Theorem 3.5. □

4. Unique Vertex Colorability

Let G be a colorable graph with proper coloring ν, and let k be the number of distinct colors in $\nu(V)$. Then G is a k-colorable graph, which has a coloring using all k colors. Before giving the definition of the polynomials $g_1, \ldots, g_n$ in Theorem 1.4, we develop some notation. Although this set of polynomials will depend on ν, for notational simplicity, we will assume that this dependency is understood. The *color class* $cl(i)$ of a vertex $i \in V$ is the set of vertices with the same color as

i. The *representative* of a color class is the largest vertex contained in it. We set $m_1 < m_2 < \cdots < m_k = n$ to be the representatives of the k color classes.

For a subset $U \subseteq V$ of the vertices, let h_U^d be the sum of all monomials of degree d in the indeterminates $\{x_i : i \in U\}$. We also set $h_U^0 = 1$. For each vertex $i \in V$, we define a polynomial g_i as follows:

$$g_i = \begin{cases} x_i^k - 1 & \text{if } i = m_k, \\ h^j_{\{m_j,\ldots,m_k\}} & \text{if } i = m_j \text{ for some } j, \\ h^1_{\{i,m_2,\ldots,m_k\}} & \text{if } i \in cl(m_1), \\ x_i - x_{\max cl(i)} & \text{otherwise.} \end{cases} \tag{4.1}$$

One should think (loosely) of the first case of (4.1) as corresponding to a choice of a color for the last vertex; the second and third, to subsets of vertices in different color classes; and the fourth, to the fact that elements in the same color class should have the same color. These polynomials encode the coloring ν algebraically in a computationally useful way (see Lemmas 4.2 and 4.4 below).

Example 4.1. In Example 1.7, the displayed coloring gives us $(m_1, m_2, m_3) = (10, 11, 12)$ and the polynomials from the second and third cases of (4.1) are

$$\begin{aligned} g_1 &= h^1_{\{1,11,12\}} = x_1 + x_{11} + x_{12}, \\ g_5 &= h^1_{\{5,11,12\}} = x_5 + x_{11} + x_{12}, \\ g_8 &= h^1_{\{8,11,12\}} = x_8 + x_{11} + x_{12}, \\ g_{10} &= h^1_{\{10,11,12\}} = x_{10} + x_{11} + x_{12}, \\ g_{11} &= h^2_{\{11,12\}} = x_{11}^2 + x_{11}x_{12} + x_{12}^2. \end{aligned}$$

One can also easily check that the other polynomials from Example 1.7 arise from the other cases of (4.1). □

Recall that a *reduced Gröbner basis* $\mathcal{G}$ is a Gröbner basis such that (1) the coefficient of $in_\prec(g)$ for each $g \in \mathcal{G}$ is 1 and (2) the leading monomial of any $g \in \mathcal{G}$ does not divide any monomial occurring in another polynomial in $\mathcal{G}$. Given a term order, reduced Gröbner bases exist and are unique.

Lemma 4.2. *Let $\prec$ be the lexicographic ordering induced by $x_n \prec x_{n-1} \prec \cdots \prec x_1$. Then the set of polynomials $\{g_1, \ldots, g_n\}$ is the reduced Gröbner basis with respect to $\prec$ for the ideal it generates.*

Proof. It is clear by construction that the initial terms of $\{g_1, \ldots, g_n\}$ are relatively prime. It follows that these polynomials form a Gröbner basis for the ideal they generate (again by some reference). That they are reduced also follows by inspection of (4.1). □

The following innocuous-looking fact is a very important ingredient in the proof of Theorem 1.4.

Lemma 4.3. *Let U be a subset of the vertices of G, and suppose that $\{i, j\} \subseteq U$. Then, for all nonnegative integers d,*

$$(x_i - x_j)h_U^d = h^{d+1}_{U \setminus \{j\}} - h^{d+1}_{U \setminus \{i\}}. \tag{4.2}$$

Proof. We induct on the number of elements in U. When $U = \{i, j\}$, the relation is clear from

$$h^d_{\{i,j\}} = \frac{x_i^{d+1} - x_j^{d+1}}{x_i - x_j}.$$

Suppose now that U has at least three elements and let $l \in U$ be different from i and j. Then,

$$\begin{aligned}
(x_i - x_j)h^d_U &= (x_i - x_j)\sum_{r=0}^{d} x_l^r h^{d-r}_{U\setminus\{l\}} \\
&= \sum_{r=0}^{d} x_l^r (x_i - x_j) h^{d-r}_{U\setminus\{l\}} \\
&= \sum_{r=0}^{d} x_l^r \left(h^{d+1-r}_{U\setminus\{j,l\}} - h^{d+1-r}_{U\setminus\{i,l\}} \right) \\
&= \sum_{r=0}^{d} x_l^r h^{d+1-r}_{U\setminus\{j,l\}} - \sum_{r=0}^{d} x_l^r h^{d+1-r}_{U\setminus\{i,l\}} \\
&= \left(h^{d+1}_{U\setminus\{j\}} - x_l^{d+1} \right) - \left(h^{d+1}_{U\setminus\{i\}} - x_l^{d+1} \right) \\
&= h^{d+1}_{U\setminus\{j\}} - h^{d+1}_{U\setminus\{i\}}.
\end{aligned}$$

This completes the induction and the proof. □

That the polynomials $g_1, \ldots, g_n$ represent an algebraic encoding of the coloring ν is explained by the following lemma.

Lemma 4.4. *Let $g_1, \ldots, g_n$ be given as in (4.1). Then the following three properties hold for the ideal $A = \langle g_1, \ldots, g_n \rangle$:*

1. $I_{G,k} \subseteq A$,
2. *A is radical,*
3. $|V(A)| = k!$.

Proof. First assume that $I_{G,k} \subseteq A$. Then A is radical from Lemma 2.1. Moreover, since the polynomials $\{g_1, \ldots, g_n\}$ form a Gröbner basis for the ideal A, the number of standard monomials of A is equal to $|V(A)|$. By inspection of (4.1) using the ordering in Lemma 4.2, we have $|\mathcal{B}_{\prec}(A)| = k!$, and therefore (3) is proved.

We now prove statement (1). First, we give explicit representations of polynomials $x_i^k - 1 \in I_{n,k}$ in terms of the generators of A. We first claim that for $i = 1, \ldots, k$, we have

$$x^k_{m_i} - 1 = x^k_n - 1 + \sum_{l=i}^{k-i} \left[\prod_{j=l+1}^{k} \left(x_{m_i} - x_{m_j} \right) \right] h^l_{\{m_l,\ldots,m_k\}}. \tag{4.3}$$

To verify (4.3), we will use Lemma 4.3 and induction to prove that for each $s \leq k - i$, the sum on the right hand side above is equal to

$$\prod_{j=s+i}^{k} \left(x_{m_i} - x_{m_j} \right) h^{s+i-1}_{\{m_i, m_{s+i},\ldots,m_k\}} + \sum_{l=s+i}^{k-i} \left[\prod_{j=l+1}^{k} \left(x_{m_i} - x_{m_j} \right) \right] h^l_{\{m_l,\ldots,m_k\}}. \tag{4.4}$$

For $s = 1$, this is clear. In general, using Lemma 4.3, the first term on the left hand side of (4.4) is

$$\prod_{j=s+i+1}^{k} (x_{m_i} - x_{m_j}) \left(h^{s+i}_{\{m_i, m_{s+1+i}, \ldots, m_k\}} - h^{s+i}_{\{m_{s+i}, \ldots, m_k\}} \right),$$

which is easily seen to cancel the first summand in the sum found in (4.4). Now, equation (4.4) with $s = k - i$ gives us that the right hand side of (4.3) is

$$x_n^k - 1 + (x_{m_i} - x_{m_k}) h^{k-1}_{\{m_i, m_k\}} = x_n^k - 1 + x_{m_i}^k - x_n^k = x_{m_i}^k - 1,$$

proving the claim.

It remains to show that $x_i^k - 1 \in A$ for all $i \in V$. We first claim that for those $i \in V$ not in $\{m_1, \ldots, m_k\}$, we have $x_i - x_{m_i} \in A$. For those i not in the color class of vertex m_1, this is clear from (4.1). Otherwise,

$$g_i - g_{m_1} = h^1_{\{i, m_2, \ldots, m_k\}} - h^1_{\{m_1, \ldots, m_k\}} = x_i - x_{m_1} \in A,$$

as desired. For $i \notin \{m_1, \ldots, m_k\}$, let $f_i = x_i - x_{m_i}$ and notice that

$$x_{m_i}^k - 1 = (x_i - f_i)^k - 1 = x_i^k - 1 + f_i h \in A$$

for some polynomial h. It follows that $x_i^k - 1 \in A$.

Finally, we must verify that the other generators of $I_{G,k}$ are in A. To accomplish this, we will prove the following stronger statement:

$$U \subseteq \{m_1, \ldots, m_k\} \text{ with } |U| \geq 2 \implies h_U^{k+1-|U|} \in A. \tag{4.5}$$

We downward induct on $s = |U|$. In the case $s = k$, we have $U = \{m_1, \ldots, m_k\}$. But then as is easily checked $g_{m_1} = h_U^{k+1-|U|} \in A$. For the general case, we will show that if one polynomial $h_U^{k+1-|U|}$ is in A, with $|U| = s < k$, then $h_U^{k+1-|U|} \in A$ for any subset $U \subseteq \{m_1, \ldots, m_k\}$ of cardinality s. In this regard, suppose that $h_U^{k+1-|U|} \in A$ for a subset U with $|U| = s < k$. Let $u \in U$ and $v \in \{m_1, \ldots, m_k\} \backslash U$, and examine the following equality (using Lemma 4.3):

$$(x_u - x_v) h^{k-s}_{U \cup \{v\}} = h_U^{k-s+1} - h^{k-s+1}_{U \cup \{v\} \backslash \{u\}}.$$

By induction, the left hand side of this equation is in A and therefore the assumption on U implies that

$$h^{k-s+1}_{U \cup \{v\} \backslash \{u\}} \in A.$$

This shows that we may replace any element of U with any element of $\{m_1, \ldots, m_k\}$. Since there is a subset U of size s with $h_U^{k+1-|U|} \in A$ (see (4.1)), it follows from this that we have $h_U^{k+1-|U|} \in A$ for any subset U of size s. This completes the induction.

A similar trick as before using polynomials $x_i - x_{m_i} \in A$ proves that we may replace in (4.5) the requirement that $U \subseteq \{m_1, \ldots, m_k\}$ with one that says that U consists of vertices in different color classes. If $\{i, j\} \in E$, then i and j are in different color classes, and therefore the generator $h^{k-1}_{\{i,j\}} \in I_{G,k}$ is in A. This finishes the proof of the lemma. □

Remark 4.5. Property (1) in the lemma says that $V(A)$ contains proper colorings of G while properties (2) and (3) say that, up to permutation of the colors, the zeroes of the polynomials $g_1, \ldots, g_n$ correspond to the single proper coloring given by ν.

Lemma 4.6. *Suppose that G is uniquely k-colorable. Then the following two statements hold:*

(1) *If $\{i,j\} \subseteq V$ have the same color, then $x_i - x_j \in I_{G,k}$.*
(2) *If $U \subseteq V$ is a set with $|U| \geq 2$ consisting of vertices with all different colors, then $h_U^{k+1-|U|} \in I_{G,k}$.*

Proof. Let $\mathbf{v} = (v_1, \ldots, v_n) \in V(I_{G,k})$, which by Lemma 3.1 corresponds to a proper coloring. Since G is uniquely k-colorable, it follows that $v_i - v_j = 0$ for each i and j in the same color class. Thus $x_i - x_j \in I(V(I_{G,k})) = I_{G,k}$ since $I_{G,k}$ is radical. To prove the second statement, we induct on the size of U. Suppose that $U = \{i,j\}$ consists of two vertices with different colors, and let $\mathbf{v} = (v_1, \ldots, v_n) \in V(I_{G,k})$. Then by Lemma 4.3,

$$(v_i - v_j)h_U^{k+1-|U|}(\mathbf{v}) = h_{U\setminus\{j\}}^k(\mathbf{v}) - h_{U\setminus\{i\}}^k(\mathbf{v}) = v_i^k - v_j^k = 0.$$

Since $v_i \neq v_j$, it follows that $h_U^{k+1-|U|} \in I_{G,k}$ in this case (using as before that $I_{G,k}$ is radical). For $|U| > 2$, we have,

$$(v_i - v_j)h_U^{k+1-|U|}(\mathbf{v}) = h_{U\setminus\{j\}}^{k+1-|U\setminus\{j\}|}(\mathbf{v}) - h_{U\setminus\{i\}}^{k+1-|U\setminus\{i\}|}(\mathbf{v}) = 0 - 0 = 0,$$

by Lemma 4.3 and the induction hypothesis. Again, it follows that $h_U^{k+1-|U|}(\mathbf{v}) = 0$, completing the induction and the proof. □

Before proving our main theorem, we define the g in Theorem 1.4 using a "dual" set of auxiliary polynomials $\overline{g}_1, \ldots, \overline{g}_n$. Given a subset $U \subseteq V$ of the vertices of G, we let K_U denote the graph on vertices V with a clique on vertices U and isolated other vertices. For $i = 1, \ldots, n$, set

$$\overline{g}_i = \begin{cases} 1 & \text{if } i = m_k, \\ f_{K_{\{m_j,\ldots,m_k\}}} & \text{if } i = m_j \text{ for some } j, \\ f_{K_{\{i,m_2,\ldots,m_k\}}} & \text{if } i \in cl(m_1), \\ h_{\{i,\max cl(i)\}}^{k-1} & \text{otherwise.} \end{cases} \tag{4.6}$$

We can now define

$$g = \overline{g}_1 \cdots \overline{g}_n. \tag{4.7}$$

The following is a duality relationship between $g_1, \ldots, g_n$ and g.

Lemma 4.7. $I_{n,k} : \langle g_1, \ldots, g_n \rangle = I_{n,k} + \langle g \rangle$.

Proof. Since all the ideals in consideration are radical by Lemmas 2.1 and 4.4, equation (2.1) says that we need to show:

$$V(I_{n,k} + \langle g \rangle) = V(I_{n,k}) \setminus V(\langle g_1, \ldots, g_n \rangle).$$

First, suppose that $\mathbf{v} = (v_1, \ldots, v_n)$ is contained in the left-hand side of the above equation; we will verify it is in the right-hand side. In this case, $\overline{g}_i(\mathbf{v}) = 0$ for some i. Suppose that i arises from the fourth case of (4.6), and let j be such that $m_j = \max cl(i)$. If $v_i = v_{m_j}$, then $h_{\{i,m_j\}}^{k-1}(\mathbf{v}) = kv_i^{k-1} \neq 0$, a contradiction. It follows that $v_i \neq v_{m_j}$, and therefore $\mathbf{v} \notin V(\langle g_1, \ldots, g_n \rangle)$. If i comes from cases two or three in (4.6), then $\overline{g}_i(\mathbf{v}) = 0$ says that two coordinates of $\mathbf{v}$ that represent vertices in different color classes are equal. But then this point cannot be in $V(\langle g_1, \ldots, g_n \rangle)$ by Lemma 4.4 (specifically, Remark 4.5).

Conversely, suppose that $\mathbf{v}$ is a coloring not contained in $V(\langle g_1,\ldots,g_n\rangle)$. Then $g_i(\mathbf{v}) \neq 0$ for some i. This i cannot come from the first case of (4.1). If it arises from the fourth case, then $v_i - v_{m_j} \neq 0$ for some j. Thus, the equality $(v_i - v_{m_j})h^{k-1}_{\{i,m_j\}}(\mathbf{v}) = v_i^k - v_{m_j}^k = 0$ implies that $\overline{g}_i(\mathbf{v}) = 0$, as desired. Finally, suppose that $g_i(\mathbf{v}) \neq 0$ for i from cases two or three, and let $S = \{i, m_2, \ldots, m_k\}$ or $S = \{m_j, \ldots, m_k\}$, correspondingly. Consider all subsets $U \subseteq S$ with at least 2 elements such that $h_U^{k+1-|U|}(\mathbf{v}) \neq 0$ and choose one of minimum cardinality; this set exists by assumption. If $U = \{i,j\}$, then $(v_i - v_j)h_U^{k+1-|U|}(\mathbf{v}) = v_i^k - v_j^k = 0$ so that $v_i = v_j$ and $\overline{g}_i(\mathbf{v}) = 0$. Otherwise, if $\{i,j\} \subseteq U$, then

$$(v_i - v_j)h_U^{k+1-|U|}(\mathbf{v}) = h_{U\setminus\{j\}}^{k+1-|U\setminus\{i\}|}(\mathbf{v}) - h_{U\setminus\{i\}}^{k+1-|U\setminus\{i\}|}(\mathbf{v}) = 0,$$

by Lemma 4.3 and the minimality of U. Again, it follows that $v_i = v_j$ and $\overline{g}_i(\mathbf{v}) = 0$. Therefore, in all cases, $\mathbf{v} \in V(I_{n,k} + \langle g\rangle)$. This finishes the proof. □

We are now in a position to prove our main theorem.

Proof of Theorem 1.4. (1) $\Rightarrow$ (2): Suppose the graph G is uniquely k-colorable and construct the set of g_i from (4.1); we will prove that $g_i \in I_{G,k}$ for each $i \in V$. By Lemma 4.6, polynomials of the form $x_i - x_{m_i}$ are in $I_{G,k}$, and by definition of $I_{G,k}$, we have $x_n^k - 1 \in I_{G,k}$. Finally, since the sets $U = \{m_j, \ldots, m_k\}$ and $V = \{i, m_2, \ldots, m_k\}$ consist of vertices with different colors, those g_i of the form h_U^j and h_V^1 are in $I_{G,k}$ again by Lemma 4.3.

(2) $\Rightarrow$ (3): Suppose that $A = \langle g_1, \ldots, g_n\rangle \subseteq I_{G,k}$. From Lemmas 3.3 and 4.7, we have

$$\begin{aligned} I_{n,k} + \langle f_G\rangle &= I_{n,k} : I_{G,k} \\ &\subseteq I_{n,k} : A \\ &= I_{n,k} + \langle g\rangle. \end{aligned}$$

This proves that $f_G \in I_{n,k} + \langle g\rangle$.

(3) $\Rightarrow$ (1): Assume that $f_G \in I_{n,k} + \langle g\rangle$. Then, $I_{n,k} : I_{G,k} \subseteq I_{n,k} + \langle g\rangle = I_{n,k} : A$. Applying Lemmas 2.1 and 4.4, we have

$$k^n - k! = |V(I_{n,k})\setminus V(A)| = |V(I_{n,k} : A)| \leq |V(I_{n,k} : I_{G,k})| \leq k^n - k!, \tag{4.8}$$

since the number of improper colorings is at most $k^n - k!$. It follows that equality holds throughout (4.8) so that the number of proper colorings is $k!$. Therefore, G is uniquely k-colorable, completing the proof. □

Collecting the results of this section, we can now also prove Theorem 1.6 from the introduction.

Proof of Theorem 1.6. By Lemma 4.2, it is enough to show that $A = \langle g_1, \ldots, g_n\rangle = I_{G,k}$. From Lemma 4.4, we know that $I_{G,k} \subseteq A$, and the other inclusion is clear from the equivalence of (1) and (2) in Theorem 1.4. □

5. Algorithms and Xu's Conjecture

In this section we describe the algorithms implied by Theorems 1.1 and 1.4, and illustrate their usefulness by disproving a conjecture of Xu.[1]

[1] Code that performs this calculation along with an implementation of the algorithms in this section can be found **here**.

First, from Theorem 1.1, we have the following methods for determining k-colorability.

Algorithm 5.1.
Input: A graph G with vertices $V = \{1, \dots, n\}$ and edges E, and a positive integer k.
Output: ***true*** if G is k-colorable; otherwise ***false***.

Method 1:
(1) Compute a Gröbner basis $\mathcal{G}$ of $I_{G,k}$.
(2) Compute the normal form of the constant polynomial 1 wrt. $\mathcal{G}$.
(3) Return ***false*** if the normal form is zero; otherwise return ***true***.

Method 2:
(1) Set $\mathcal{G} := \{x_i^k - 1 : i \in V\}$.
(2) Set $f := 1$.
(3) For $\{i, j\} \in E$:
Compute the normal form g of $(x_i - x_j)f$ wrt. $\mathcal{G}$, and set $f := g$.
(4) Return ***false*** if f is zero; otherwise return ***true***.

Method 3:
(1) Set $\mathcal{G} := \{f_H : H \in \mathcal{H}\}$, where $\mathcal{H}$ is the set of graphs with vertices $\{1, \dots, n\}$ consisting of a clique of size $k + 1$ and isolated vertices.
(2) Set $f := 1$.
(3) For $\{i, j\} \in E$:
Compute the normal form g of $(x_i - x_j)f$ wrt. $\mathcal{G}$, and set $f := g$.
(4) Return ***false*** if f is zero; otherwise return ***true***.

The analogue of this algorithm for unique colorability is given by Theorem 1.4.

Algorithm 5.2.
Input: A graph G with vertices $V = \{1, \dots, n\}$ and edges E, and a k-coloring of G.
Output: ***true*** if G is uniquely k-colorable; otherwise ***false***.

Method 1:
(1) Compute a Gröbner basis $\mathcal{G}$ of $I_{G,k}$.
(2) For $i \in V$:
Compute the normal form of g_i wrt. $\mathcal{G}$.
Return ***false*** if the normal form is nonzero.
(3) Return ***true***.

Method 2:
(1) Compute a Gröbner basis $\mathcal{G}$ of $I_{n,k} + \langle g \rangle$.
(2) Set $f := 1$.
(3) For $\{i, j\} \in E$:
Compute the normal form g of $(x_i - x_j)f$ wrt. $\mathcal{G}$, and set $f := g$.
(4) Return ***true*** if f is zero; otherwise return ***false***.

In [13], Xu showed that if G is a uniquely k-colorable graph with $|V| = n$ and $|E| = m$, then $m \geq (k-1)n - \binom{k}{2}$, and this bound is best possible. He went on to conjecture that if G is uniquely k-colorable with $|V| = n$ and $|E| = (k-1)n - \binom{k}{2}$, then G contains a k-clique. In [1], this conjecture was shown to be false for $k = 3$ and $|V| = 24$ using the graph in Figure 2; however, the verification is somewhat complicated. We prove that this graph is a counterexample to Xu's conjecture using Algorithm 5.2, Method 1. The computation requires approximately an hour of processor time on a laptop PC; the code can be downloaded from the link at the beginning of this section.

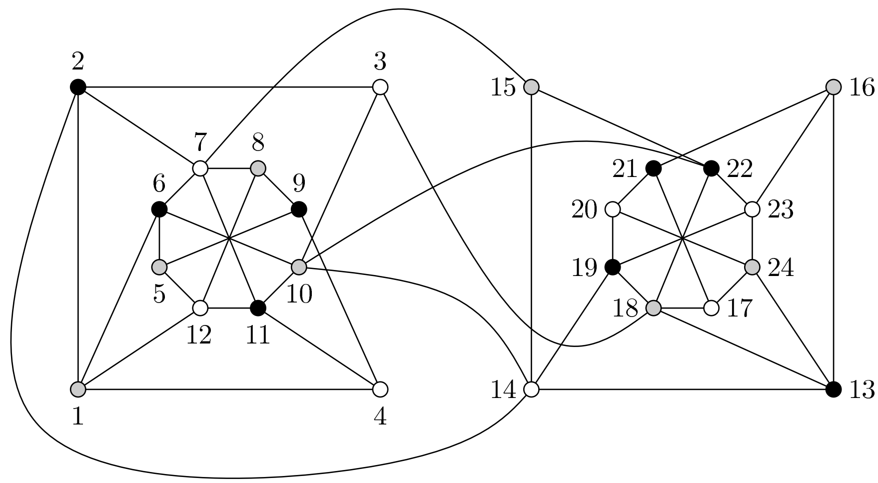

FIGURE 2. A counterexample to Xu's conjecture

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