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Integrating Factors for Second-order ODEs

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A systematic algorithm for building integrating factors of the form $\mu(x, y)$, $\mu(x, y')$ or $\mu(y, y')$ for second-order ODEs is presented. The algorithm can determine the existence and explicit form of the integrating factors themselves without solving any differential equations, except for a linear ODE in one subcase of the $\mu(x, y)$ problem. Examples of ODEs not having point symmetries are shown to be solvable using this algorithm. The scheme was implemented in Maple, in the framework of the *ODEtools* package and its ODE-solver. A comparison between this implementation and other computer algebra ODE-solvers in tackling non-linear examples from Kamke's book is shown.
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1. Introduction

Although in principle it is always possible to determine whether a given ODE is exact (a total derivative), there is no known method which is always successful in making arbitrary ODEs exact. For nth order ODEs, as in the case of symmetries, integrating factors (μ) are determined as solutions of an nth order linear PDE in $n+1$ variables, and to solve this determining PDE is a major problem in itself.

Despite the fact that the determining PDE for μ naturally splits into a PDE system, the problem is, as a whole, too general, and to solve it a restriction of the problem in the form of a more concrete ansatz for μ is required. For example, in a recent work by Anco and Bluman (1997), the authors explore possible ansätze depending on the given ODE, which are useful when this ODE has known symmetries of certain type. In another work, Wolf *et al.* (1999) explores the use of computer algebra to try various ansätze for μ, no matter the ODE input, but successively increasing the order of the derivatives (up to the nth $-$ 1 order) on which μ depends; the idea is to try to maximize the splitting so as to increase the chances of solving the resulting PDE system by first simplifying it using differential Gröbner basis techniques.

Bearing this in mind, this paper presents a method, for second-order explicit ODEs,[§] which systematically determines the existence and the explicit form of integrating factors when they depend on only two variables; that is, when they are of the form $\mu(x, y)$,

[§]We say that a second-order ODE is in explicit form when it appears as $y'' - \Phi(x, y, y') = 0$. Also, we exclude from the discussion the case of a linear ODE and an integrating factor of the form $\mu(x)$, already known to be the solution to the adjoint ODE.

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$\mu(x, y')$ or $\mu(y, y')$. The approach works without solving any auxiliary differential equations, except for a linear ODE in one subcase of the $\mu(x, y)$ problem, and is based on the use of the forms of the ODE families admitting such integrating factors. It turns out that with this restriction, μ depends on only two variables, the use of differential Gröbner basis techniques is not necessary; these integrating factors, when they exist, can be given directly by identifying the input ODE as a member of one of various related ODE families.

The exposition is organized as follows. In Section 2, the standard formulation of the determination of integrating factors is briefly reviewed and the method we used for obtaining the aforementioned integrating factors $\mu(x, y)$, $\mu(x, y')$ or $\mu(y, y')$ is presented. In Section 3, some aspects of the integrating factor and symmetry approaches are discussed, and their complementariness is illustrated with two ODE families not having point symmetries. Section 4 contains some statistics concerning the new solving method and the second-order non-linear ODEs found in Kamke's (1959) book, as well as a comparison of performances of some popular computer algebra packages in solving a related subset of these ODEs. Finally, the conclusions contain some general remarks about the work.

Aside from this, in the Appendix, a table containing extra information concerning integrating factors for some of Kamke's ODEs is presented.

2. Integrating Factors and ODE Patterns

In this paper we use the term "integrating factor" in connection with the explicit form of an nth order ODE

$$y^{(n)} - \Phi(x, y, y', \ldots, y^{(n-1)}) = 0 \tag{2.1}$$

so that $\mu(x, y, y', \ldots, y^{(n-1)})$ is an integrating factor if

$$\mu(y^{(n)} - \Phi) = \frac{d}{dx} R(x, y, y', \ldots, y^{(n-1)}) \tag{2.2}$$

for some function R. The knowledge of μ is, in principle, enough to determine R by using standard formulas (see, for example Murphy's (1960) book). To determine μ, one can try to solve for it in the exactness condition, obtained by applying Euler's operator to the total derivative $H \equiv \mu(y^{(n)} - \Phi)$:

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial H}{\partial y''}\right) + \cdots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial H}{\partial y^{(n)}}\right) = 0. \tag{2.3}$$

Equation (2.3) is of the form

$$A(x, y, y', \ldots, y^{(2n-3)}) + y^{(2n-2)} B(x, y, y', \ldots, y^{(n-1)}) = 0 \tag{2.4}$$

where A is of degree $n - 1$ in $y^{(n)}$ and linear in $y^{(k)}$ for $n < k \leq (2n - 3)$, so that equation (2.4) can be split into a PDE system for μ. In the case of second-order ODEs, the subject of this work, equation (2.3) is of the form

$$A(x, y, y') + y'' B(x, y, y') = 0 \tag{2.5}$$

and the PDE system is obtained by taking A and B equal to zero:†

$$A \equiv (y'\mu_{y'y} - \mu_y + \mu_{y'x})\Phi + (\Phi_{y'x} + y'\Phi_{y'y} - \Phi_y)\mu + {y'}^2\mu_{yy}$$
$$+(\mu_y\Phi_{y'} + \mu_{y'}\Phi_y + 2\mu_{xy})y' + \mu_{y'}\Phi_x + \mu_x\Phi_{y'} + \mu_{xx} = 0 \tag{2.6}$$
$$B \equiv y'\mu_{y'y} + \Phi\mu_{y'y'} + \mu\Phi_{y'y'} + 2\mu_y + 2\mu_{y'}\Phi_{y'} + \mu_{y'x} = 0. \tag{2.7}$$

Regarding the solvability of these equations, unless a more concrete ansatz for $\mu(x, y, y')$ is given, the problem is in principle as difficult as solving the original ODE. We then studied the solution for μ of equations (2.6) and (2.7) when μ depends only on two variables; that is, for $\mu(x, y)$, $\mu(x, y')$ and $\mu(y, y')$. Concretely, we searched for the existence conditions for such integrating factors, expressed as a set of equations in Φ, plus an algebraic expression for μ as a function of Φ, valid when the existence conditions hold. Formulating the problem in that manner and taking into account the integrability conditions of the system, equations (2.6) and (2.7) turned out to be solvable for $\mu(x, y)$, but appeared to us untractable when μ depends on two variables, one of which is y'.

We then considered a different approach, taking into account from the beginning the form of the ODE family admitting a given integrating factor. As shown in the following sections, it turns out that, using that piece of information (equation (2.11) below), when μ depends only on two variables, the existence conditions and the integrating factors themselves can be systematically determined; and in the cases $\mu(x, y')$ and $\mu(y, y')$, this can be done without solving any differential equations.

Concerning the ODE families admitting given integrating factors, we note that, from equation (2.2)

$$\mu(x, y, y', \ldots, y^{(n-1)}) = \frac{\partial R}{\partial y^{(n-1)}} \tag{2.8}$$

and hence, the first integral R is of the form

$$R = G(x, y, \ldots, y^{(n-2)}) + \int \mu dy^{(n-1)} \tag{2.9}$$

for some function G. In turn, since R is a first integral, it satisfies

$$R_x + y'R_y + \cdots + \Phi R_{y^{(n-1)}} = 0. \tag{2.10}$$

Inserting equation (2.9) into the above and solving for $y^{(n)}$ leads to the general form of an ODE admitting a given integrating factor:

$$y^{(n)} = \frac{-1}{\mu}\left[\frac{\partial}{\partial x}\left(\int \mu\, dy^{(n-1)} + G\right) + \cdots + y^{(n-1)}\frac{\partial}{\partial y^{(n-2)}}\left(\int \mu\, dy^{(n-1)} + G\right)\right]. \tag{2.11}$$

2.1. SECOND-ORDER ODEs AND INTEGRATING FACTORS OF THE FORM $\mu(x, y)$

We consider first the determination of integrating factors of the form $\mu(x, y)$, which turns out to be straightforward.‡ The determining equations (2.6) and (2.7) for this case

†In a recent work by Anco and Bluman (1997), the authors arrive at equations (2.5) and (2.7) departing from the adjoint linearized system corresponding to a given ODE; the possible splitting of equation (2.4) into an overdetermined system for μ is also mentioned. However, in that work, y'' of equation (2.5) above appears replaced by $\Phi(x, y, y')$, and the authors discuss possible alternatives to tackle equations (2.5) and (2.7) instead of equations (2.6) and (2.7).

‡The result for *Case A* presented in this subsection is also presented as Lemma 3.8 in Sheftel's (1997) book.

are given by:

$$y'^2\mu_{yy} + 2\mu_{xy}y' + \mu\Phi_{y'x} + \mu\Phi_{y'y}y' - \mu\Phi_y - \mu_y\Phi + \mu_y y'\Phi_{y'} + \mu_{xx} + \mu_x\Phi_{y'} = 0$$
$$\mu\Phi_{y'y'} + 2\mu_y = 0. \quad (2.12)$$

Although the use of integrability conditions is enough to tackle this problem, the solving of equations (2.12) can be directly simplified if we take into account the ODE family admitting an integrating factor $\mu(x, y)$. From equation (2.11), that ODE family takes the form

$$y'' = a(x, y)y'^2 + b(x, y)y' + c(x, y), \quad (2.13)$$

where

$$a(x, y) = -\frac{\mu_y}{\mu}, \qquad b(x, y) = -\frac{G_y + \mu_x}{\mu}, \qquad c(x, y) = -\frac{G_x}{\mu} \quad (2.14)$$

and $G(x, y)$ is an arbitrary function of its arguments. Hence, as a shortcut to solving equations (2.12), one can take equation (2.13) as an existence condition, Φ must be a polynomial of degree two in y', and directly solve equations (2.14) for μ. The calculations are straightforward; there are two different cases.

Case A: $2a_x - b_y \neq 0$. Defining the two auxiliary quantities

$$\varphi \equiv c_y - ac - b_x, \qquad \Upsilon \equiv a_{xx} + a_x b + \varphi_y \quad (2.15)$$

an integrating factor of the form $\mu(x, y)$ exists only when

$$\Upsilon_y - a_x = 0, \qquad \Upsilon_x + \varphi + b\Upsilon - \Upsilon^2 = 0 \quad (2.16)$$

and is then given in solved form, in terms of a, b and c by

$$\mu(x, y) = \exp\left(\int \left(-\Upsilon + \frac{\partial}{\partial x}\int a\, dy\right) dx - \int a\, dy\right). \quad (2.17)$$

So, in this case, when an integrating factor of this type exists there is only one[†] and it can be determined without solving any differential equations.

Case B: $2a_x - b_y = 0$. Redefining $\varphi \equiv c_y - ac$, an integrating factor of the form $\mu(x, y)$ exists only when

$$a_{xx} - a_x b - \varphi_y = 0, \quad (2.18)$$

and then $\mu(x, y)$ is given by

$$\mu(x, y) = \nu(x)\mathrm{e}^{-\int a dy} \quad (2.19)$$

where $\nu(x)$ is either one of the independent solutions of the second-order linear ODE[‡]

$$\nu'' = A(x)\nu' + B(x)\nu, \quad (2.20)$$

and

$$A(x) \equiv 2\mathcal{I} - b, \qquad B(x) \equiv \varphi + \left(\mathcal{I} - \frac{\partial}{\partial x}\right)(b - \mathcal{I}), \qquad \mathcal{I} \equiv \frac{\partial}{\partial x}\int a dy. \quad (2.21)$$

[†]We recall that if μ is an integrating factor leading to a first integral ψ, then the product $\mu F(\psi)$, where F is an arbitrary function, is also an integrating factor, which however does not lead to a first integral independent of ψ.

[‡]When the given ODE is linear, equation (2.20) is just the corresponding adjoint equation.

So in this case, to transform equation (2.19) into an explicit expression for μ we first need to solve a second-order linear ODE. When the attempt to solve equation (2.20) is successful, using each of its two independent solutions as integrating factors leads to the general solution of equation (2.13), instead of just a reduction of order.

2.2. SECOND-ORDER ODEs AND INTEGRATING FACTORS OF THE FORM $\mu(x, y')$

When the integrating factor is of the form $\mu(x, y')$, the determining equations (2.6) and (2.7) become

$$(\Phi_y y' + \Phi_x)\mu_{y'} + (-\Phi_y + \Phi_{y'x} + \Phi_{y'y} y')\mu + \mu_{xx} + \mu_x \Phi_{y'} + \mu_{y'x}\Phi = 0$$
$$\Phi\mu_{y'y'} + \mu\Phi_{y'y'} + 2\mu_{y'}\Phi_{y'} + \mu_{y'x} = 0. \quad (2.22)$$

As in the case $\mu(x, y)$, the solution we are interested in is an expression for $\mu(x, y')$ in terms of Φ, as well as existence conditions for such an integrating factor expressed as equations in Φ. However, contrary to the case $\mu(x, y)$, we did not find a way to solve the $\mu(x, y')$ problem just using integrability conditions, neither working by hand nor using the specialized computer algebra packages *diffalg* (Boulier *et al.*, 1995) and *standard_form* (Reid and Wittkopf, 1993). We then considered approaching the problem as explained in the previous subsection, departing from the form of equation (2.9) for $\mu = \mu(x, y')$:

$$y'' = \Phi(x, y, y') \equiv -\frac{F_x + G_x + G_y y'}{F_{y'}} \quad (2.23)$$

where $G(x, y)$ and $F(x, y')$ are arbitrary functions of their arguments and

$$\mu(x, y') = F_{y'}. \quad (2.24)$$

Now, equation (2.23) is not polynomial in either x, y or y', and hence, its use to simplify and solve the problem is less straightforward than in the case $\mu(x, y)$. However, in equation (2.23), all the dependence on y comes from $G(x, y)$ in the numerator, and as shown below, this fact is a key to solving the problem. Considering ODEs for which $\Phi_y \neq 0$,[†] the approach we used can be summarized in the following three lemmas whose proofs are developed separately for convenience.

LEMMA 1. *For all linear ODEs of the family equation (2.23), an integrating factor of the form $\mu(x, y')$ such that $\mu_{y'} \neq 0$, when it exists, can be determined directly from the coefficient of y in the input ODE.*

LEMMA 2. *For all non-linear ODEs of equation (2.23), the knowledge of $\mu(x, y')$ up to a factor depending on x, that is, of $\mathcal{F}(x, y')$ satisfying*

$$\mathcal{F}(x, y') = \frac{\mu(x, y')}{\tilde{\mu}(x)} \quad (2.25)$$

is enough to determine $\tilde{\mu}(x)$ by means of an integral.

LEMMA 3. *For all non-linear ODEs members of equation (2.23), it is always possible to determine a function $\mathcal{F}(x, y')$ satisfying equation (2.25).*

[†]ODEs *missing* y may also have integrating factors of the form $\mu(x, y')$. Such an ODE however can always be reduced to first order by a change of variables, so that the determination of a $\mu(x, y')$ for it is equivalent to solving a first-order ODE problem, not the focus of this work.

Corollary. *For all second-order ODEs such that $\Phi_y \neq 0$, the determination of $\mu(x, y')$ (if the ODE is linear we assume $\mu_{y'} \neq 0$), when it exists, can be performed systematically and without solving any differential equations.*

Proof of Lemma 1. For equation (2.23) to be linear and not missing y, either G_x or G_y must be linear in y. Both G_x and G_y cannot simultaneously be linear in y since, in such a case, $G_x/F_{y'}$ or $y'G_y/F_{y'}$ would be non-linear in $\{y, y'\}$;[†] therefore, either $G_{yy} = 0$ or $G_{xy} = 0$.

Case A: G_y is linear in y and $G_{xy} = 0$. Hence, G is given by

$$G = C_2 y^2 + C_1 y + g(x) \tag{2.26}$$

where $g(x)$ is arbitrary. From equation (2.23), in order to have $y'G_y/F_{y'}$ linear in $\{y, y'\}$, $F_{y'}$ must be of the form $\nu(x)y'$ for some function $\nu(x)$. Also, $F_x/F_{y'}$ can have a term linear in y', and a term proportional to $1/y'$ to cancel with the one coming from $G_x/F_{y'} = g'/F_{y'}$, so that

$$F_{y'} = \nu y', \qquad F_x = \frac{\nu' {y'}^2}{2} - g' \tag{2.27}$$

where the coefficient $\nu'/2$ in the second equation above arises from the integrability conditions between both equations. Equation (2.23) is then of the form

$$y'' = -\frac{\nu'}{2\nu} y' - \frac{2C_2}{\nu} y - \frac{C_1}{\nu} \tag{2.28}$$

and hence, a linear ODE $y'' = a(x)y' + b(x)y$ has an integrating factor $\mu(x, y') = y'/b$ when $b'/b - 2a = 0$.

Case B: G_x is linear in y and $G_{yy} = 0$. In this case, in order to have equation (2.23) linear, $F_{y'}$ cannot depend on y', so that the integrating factor is of the form $\mu(x)$ and hence the case is of no interest: we end up with the standard search for $\mu(x)$ as the solution to the adjoint of the original linear ODE.

Proof of Lemma 2. It follows from equations (2.23) and (2.24) that, given $\mathcal{F}$ satisfying (2.25),

$$\frac{\partial}{\partial y}(\Phi(x, y, y')\mathcal{F}(x, y')) = -\frac{G_{yx}(x, y) + G_{yy}(x, y)y'}{\tilde{\mu}(x)}. \tag{2.29}$$

Hence, by taking coefficients of y' in the above,

$$\begin{aligned} \varphi_1 &\equiv \Phi_y(x, y, y')\mathcal{F}(x, y') - y' \frac{\partial}{\partial y'}(\Phi_y(x, y, y')\mathcal{F}(x, y')) = -\frac{G_{yx}(x, y)}{\tilde{\mu}(x)} \\ \varphi_2 &\equiv \frac{\partial}{\partial y'}(\Phi_y(x, y, y')\mathcal{F}(x, y')) = -\frac{G_{yy}(x, y)}{\tilde{\mu}(x)} \end{aligned} \tag{2.30}$$

where the left-hand-sides can be calculated explicitly since they depend only on Φ and

[†]We are only interested in the case $\mu_{y'} = F_{y'y'} \neq 0$.

the given $\mathcal{F}$. Similarly,

$$\varphi_3 \equiv -\frac{\partial}{\partial y'}(\Phi(x,y,y')\mathcal{F}(x,y')) = \frac{F_{y'x}(x,y') + G_y(x,y)}{\tilde{\mu}(x)}$$
$$\varphi_4 \equiv \frac{\partial}{\partial y'}\mathcal{F}(x,y') = \frac{F_{y'y'}(x,y')}{\tilde{\mu}(x)}. \tag{2.31}$$

Now, since in this case the ODE family equation (2.23) is non-linear by hypothesis, either φ_2 or φ_4 is different from zero, so that at least one of the pairs $\{\varphi_1, \varphi_2\}$ or $\{\varphi_3, \varphi_4\}$ can be used to determine $\tilde{\mu}(x)$ as the solution of a first-order linear ODE. For example, if $\varphi_2 \neq 0$,

$$\frac{\partial}{\partial y}(\varphi_1(x,y)\tilde{\mu}(x)) = \frac{\partial}{\partial x}(\varphi_2(x,y)\tilde{\mu}(x)) \tag{2.32}$$

from where

$$\tilde{\mu}(x) = e^{\displaystyle\int \frac{1}{\varphi_2}\left(\frac{\partial \varphi_1}{\partial y} - \frac{\partial \varphi_2}{\partial x}\right) dx}. \tag{2.33}$$

If $\varphi_2 = 0$ then $\varphi_4 \neq 0$ and we obtain

$$\tilde{\mu}(x) = e^{\displaystyle\int \frac{1}{\varphi_4}\left(\frac{\partial \varphi_3}{\partial y'} - \frac{\partial \varphi_4}{\partial x}\right) dx}. \tag{2.34}$$

When combined with equation (2.25), equations (2.33) and (2.34) alternatively give both an explicit solution to the problem and an existence condition, since a solution $\tilde{\mu}(x)$, and hence an integrating factor of the form $\mu(x,y')$, exists if the integrand in equation (2.33) or equation (2.34) only depends on x.

Proof of Lemma 3. We start from equation (2.23) by considering the expression

$$\Upsilon \equiv \Phi_y = -\frac{G_{xy}(x,y) + G_{yy}(x,y)y'}{F_{y'}(x,y')} \tag{2.35}$$

and develop the proof below splitting the problem into different cases. For each case we show how to find $\mathcal{F}(x,y')$ satisfying equation (2.25). $\mathcal{F}$ will then lead to the required integrating factor when, in addition to the conditions explained below, the existence conditions for $\tilde{\mu}(x)$ mentioned in the previous subsection are satisfied.

Case A: G_{xy}/G_{yy} depends on y. To determine whether this is the case, we cannot just analyse the ratio G_{xy}/G_{yy} itself since it is unknown. However, from equation (2.35), in this case the factors of Υ depending on y will also depend on y', and this condition can be formulated as

$$\frac{\partial}{\partial y'}\left(\frac{\Upsilon_y}{\Upsilon}\right) \neq 0. \tag{2.36}$$

When this inequation holds, we determine $F_{y'}(x,y')$ up to a factor depending on x, that is, the required $\mathcal{F}(x,y')$, as the reciprocal of the factors of Υ which depend on y' but not y.

Example. (Kamke's, 1959, ODE 226) This ODE is presented in Kamke (1959) already in exact form, so we start by rewriting it in explicit form as

$$y'' = \frac{x^2 y y' + x y^2}{y'}. \tag{2.37}$$

We determine Υ (equation (2.35)) as

$$\Upsilon = \frac{x(xy' + 2y)}{y'}. \tag{2.38}$$

The only factor of Υ containing y is:

$$xy' + 2y \tag{2.39}$$

and since this also depends on y', $\mathcal{F}(x, y')$ is given by

$$\mathcal{F}(x, y') = y'. \tag{2.40}$$

Case B: either $G_{xy} = 0$ or $G_{yy} = 0$. When the expression formed by all the factors of Υ containing y does not contain y', in equation (2.36) we will have $\frac{\partial}{\partial y'}(\frac{\Upsilon_y}{\Upsilon}) = 0$, and it is impossible to determine *a priori* whether one of the functions $\{G_{xy}, G_{yy}\}$ is zero, or alternatively their ratio does not depend on y. We then proceed by assuming the former, build an expression for $\mathcal{F}(x, y')$ as in Case A, and check for the existence of $\tilde{\mu}(x)$ as explained in the previous subsection. If $\tilde{\mu}(x)$ exists, the problem is solved; otherwise we proceed as follows.

Case C: $G_{xy}/G_{yy} \neq 0$ and does not depend on y. In this case, neither G_{xy} nor G_{yy} is zero and their ratio is a function of just x, so that

$$\begin{aligned} G_{xy} &= v_1(x) w(x, y) \\ G_{yy} &= v_2(x) w(x, y) \end{aligned} \tag{2.41}$$

for some unknown functions $v_1(x)$ and $v_2(x)$. Equation (2.35) is then given by

$$\Upsilon = w(x, y) \frac{(v_1(x) + v_2(x) y')}{F_{y'}(x, y')} \tag{2.42}$$

for some function $w(x, y)$, which is made up of the factors of Υ depending on y and not on y'. To determine $F_{y'}(x, y')$ up to a factor depending on x, we need to determine the ratio $v_1(x)/v_2(x)$. For this purpose, from equation (2.41) we build a PDE for $G_y(x, y)$,

$$G_{xy} = \frac{v_1(x)}{v_2(x)} G_{y,y}. \tag{2.43}$$

The general solution of equation (2.43) is

$$G_y = \mathcal{G}(y + p(x)) \tag{2.44}$$

where $\mathcal{G}$ is an arbitrary function of its argument and for convenience we introduced

$$p'(x) \equiv v_1(x)/v_2(x). \tag{2.45}$$

We now determine $p'(x)$ as follows. Taking into account equation (2.41),

$$v_2(x) w(x, y) = \mathcal{G}'(y + p(x)). \tag{2.46}$$

By taking the ratio between this expression and its derivative w.r.t. y we obtain

$$\mathcal{H}(y + p(x)) \equiv \frac{\partial \ln(w)}{\partial y} = \frac{\mathcal{G}''(y + p(x))}{\mathcal{G}'(y + p(x))} \tag{2.47}$$

that is, a function of $y + p(x)$ only, which we can determine since we know $w(x, y)$. If $\mathcal{H}' \neq 0$, $p'(x)$ is given by

$$p'(x) = \frac{\mathcal{H}_x}{\mathcal{H}_y} = \frac{w_{xy}w - w_x w_y}{w_{yy}w - {w_y}^2}. \tag{2.48}$$

In summary, the conditions for this case are

$$\Upsilon_y \neq 0, \qquad \frac{\partial}{\partial y'}\left(\frac{\Upsilon_y}{\Upsilon}\right) = 0, \qquad \frac{\partial^2}{\partial y \partial x}\ln(w) \neq 0, \qquad \frac{\partial^2}{\partial y \partial y}\ln(w) \neq 0 \tag{2.49}$$

and then, from equation (2.42), $\mathcal{F}(x, y')$ is given by

$$\mathcal{F}(x, y') = \frac{(p' + y')w}{\Upsilon} \tag{2.50}$$

where at this point Υ, $w(x, y)$ and $p'(x)$ are all known.

Example. (Kamke's, 1959, ODE 136) We begin by writing the ODE in explicit form as

$$y'' = \frac{h(y')}{x - y}. \tag{2.51}$$

This example is interesting since the standard search for point symmetries is made difficult by the presence of an arbitrary function of y'. Υ (equation (2.35)) is determined as

$$\Upsilon = -\frac{h(y')}{(x - y)^2} \tag{2.52}$$

and $w(x, y)$ as

$$w(x, y) = \frac{1}{(x - y)^2}. \tag{2.53}$$

Then $\mathcal{H}(y + p(x))$ (equation (2.47)) becomes

$$\mathcal{H} = \frac{2}{x - y} \tag{2.54}$$

and hence, from equation (2.48), $p'(x)$ is

$$p'(x) = -1, \tag{2.55}$$

so from equation (2.50):

$$\mathcal{F}(x, y') = \frac{1 - y'}{h(y')}. \tag{2.56}$$

Case D: $\mathcal{H} = 0$. We now discuss how to obtain $p'(x)$ when $\mathcal{H}'(y + p(x)) = 0$. We consider first the case in which $\mathcal{H} = 0$. Then, $\mathcal{G}'' = 0$ and the condition for this case is

$$\Upsilon_y = 0. \tag{2.57}$$

Recalling equation (2.44), G is given by

$$G(x, y) = C_1(y + p(x))^2 + C_2(y + p(x)) + g(x) \tag{2.58}$$

for some function $g(x)$ and some constants C_1, C_2. From equation (2.23), $\Phi(x,y,y')$ takes the form

$$\Phi(x,y,y') = -\frac{F_x(x,y') + g'(x) + (2C_1(y+p(x)) + C_2)(y'+p'(x))}{F_{y'}(x,y')}. \tag{2.59}$$

We now determine $p'(x)$ as follows. First, from the knowledge of Υ and Φ we build the two explicit expressions:

$$\Lambda \equiv \frac{1}{\Upsilon} = -\frac{F_{y'}}{2C_1(y'+p'(x))} \tag{2.60}$$

and

$$\Psi \equiv \frac{\Phi(x,y,y')}{\Upsilon} - y = \frac{F_x + g'(x)}{2C_1(y'+p'(x))} + p(x) + \frac{C_2}{2C_1}. \tag{2.61}$$

From equations (2.60) and (2.61) Λ and Ψ are related by:

$$\frac{\partial}{\partial x}((y'+p'(x))\Lambda) + \frac{\partial}{\partial y'}((y'+p'(x))\Psi) = p(x) + \frac{C_2}{2C_1} \tag{2.62}$$

where the only unknowns are $p(x)$, C_1, and C_2. By differentiating the equation above w.r.t. y' and x we obtain two equations where the only unknown is $p'(x)$:

$$\Lambda_{y'}p''(x) + (\Lambda_{xy'} + \Psi_{y'y'})(y'+p'(x)) + \Lambda_x + 2\Psi_{y'} = 0 \tag{2.63}$$

$$\Lambda p'''(x) + (\Lambda_{xx} + \Psi_{y'x})(y'+p'(x)) + (\Lambda_x + \Psi_{y'})p''(x) + \Psi_x = p'(x) \tag{2.64}$$

from which we obtain $p'(x)$ by solving a linear algebraic equation built by eliminating $p''(x)$ between equations (2.63) and (2.64).† Also, as a shortcut, if $(\Lambda_{xy'} + \Psi_{y'y'})/\Lambda_{y'}$ depends on y', then we can build a linear algebraic equation for $p'(x)$ by solving for $p''(x)$ in equation (2.63) and differentiating w.r.t. y'.

REMARK. If equation (2.63) depends neither on $p'(x)$ nor on $p''(x)$ this scheme will not succeed. However, in that case the original ODE is actually linear and given by equation (2.28). To see this, we set to zero the coefficients of $p'(x)$ and $p''(x)$ in equation (2.63), obtaining:

$$\Lambda_{y'} = \Lambda_{xy'} + \Psi_{y'y'} = \Lambda_x + 2\Psi_{y'} = 0, \tag{2.65}$$

which implies that Λ is a function of x only, and then

$$\Psi_{y'y'} = 0. \tag{2.66}$$

If we now rewrite $F(x,y')$ as

$$F(x,y') = Z(x,y') - g(x) - \Lambda(y'+p')^2 C_1 \tag{2.67}$$

and introduce this expression in equation (2.60), we obtain $Z_{y'} = 0$; similarly, using this result, equation (2.61), equation (2.66) and equation (2.67) we obtain $Z_x = 0$. Hence, Z *is a constant*, and taking into account equations (2.67) and (2.59), the ODE which led us to this case is just a non-homogeneous linear ODE of the form

$$(y+p)'' + (\Lambda'(y+p)' - 2(y+p) - C_2/C_1)/2\Lambda = 0 \tag{2.68}$$

†From equation (2.60), $\Lambda \neq 0$, so that equation (2.64) always depends on $p'''(x)$, and solving equation (2.63) for $p''(x)$ and substituting twice into equation (2.64) will lead to the desired equation for $p'(x)$. If equation (2.63) depends on $p'(x)$ but not on $p''(x)$, then equation (2.63) itself is already a linear algebraic equation for $p'(x)$.

whose homogeneous part does not depend on $p(x)$:

$$y'' + \frac{\Lambda'(x)}{2\Lambda(x)} y' - \frac{y}{\Lambda(x)} = 0 \tag{2.69}$$

and as mentioned, it is the same as equation (2.28).

Example. (Kamke's, 1959, ODE 66) This ODE is given by

$$y'' = a(c + bx + y)({y'}^2 + 1)^{3/2}. \tag{2.70}$$

Proceeding as in Case A, we determine Υ, $w(x, y)$, and $\mathcal{H}(y + p(x))$ as

$$\Upsilon = a(y'^2 + 1)^{3/2}; \qquad w(x, y) = 1; \qquad \mathcal{H} = 0. \tag{2.71}$$

From the last equation we realize that we are in Case D. We determine Λ and Ψ (equations (2.60), (2.61)) as:

$$\Lambda = \frac{1}{({y'}^2 + 1)^{3/2} a}$$
$$\Psi = c + bx. \tag{2.72}$$

We then build equation (2.62) for this ODE:

$$\frac{p''(x)}{({y'}^2 + 1)^{3/2} a} + c + bx = p(x) + \frac{C_2}{2C_1}. \tag{2.73}$$

Differentiating w.r.t. y' leads to equation (2.63):

$$-3 \frac{p''(x) y'}{({y'}^2 + 1)^{5/2} a} = 0 \tag{2.74}$$

from which it follows that $p''(x) = 0$. Using this in equation (2.64) we obtain:

$$p'(x) = b \tag{2.75}$$

after which equation (2.50) becomes

$$\mathcal{F}(x, y') = \frac{y' + b}{a({y'}^2 + 1)^{3/2}}. \tag{2.76}$$

Case E: $\mathcal{H}' = 0$ and $\mathcal{H} \neq 0$. In this case $\mathcal{H}(y+p(x)) = \mathcal{G}''/\mathcal{G}' = C_1$, so $\mathcal{G}'$ is an exponential function of its argument $(y + p(x))$ and hence from equation (2.44)

$$G(x, y) = C_2 e^{(y+p(x))C_1} + (y + p(x))C_3 + g(x) \tag{2.77}$$

for some constants C_2, C_3 and some function $g(x)$. In this case one of the conditions to be satisfied is

$$\Upsilon_y = \text{constant} \neq 0 \tag{2.78}$$

and $\Phi(x, y, y')$ will be of the form

$$\Phi(x, y, y') = -\frac{F_x(x, y') + g'(x) + (C_2 C_1 e^{(y+p(x))C_1} + C_3)(y' + p'(x))}{F_{y'}(x, y')}. \tag{2.79}$$

Taking advantage of the fact that we explicitly know C_1, we build a first expression for p' by dividing $C_1^2 e^{yC_1}$ by Υ:

$$\Lambda \equiv -\frac{F_{y'}}{C_2 e^{p(x)C_1}(y' + p'(x))}. \tag{2.80}$$

We obtain a second expression for p' by multiplying Φ by Λ and subtracting $C_1 e^{C_1 y}$

$$\Psi \equiv \frac{1}{C_2 e^{p(x)C_1}} \left(\frac{F_x + g'(x)}{y' + p'(x)} + C_3 \right). \tag{2.81}$$

As in Case D, Λ and Ψ are related by

$$\frac{\partial}{\partial x}((y' + p'(x))\Lambda) + (y' + p'(x))p'(x)\Lambda C_1 + \frac{\partial}{\partial y'}((y' + p'(x))\Psi) = \frac{C_3}{C_2 e^{p(x)C_1}} \tag{2.82}$$

where the only unknowns are C_2, C_3 and $p(x)$. Differentiating equation (2.82) w.r.t. y' we have

$$\begin{aligned}(p''(x) + p'(x)^2 C_1)\Lambda_{y'} + p'(x)(y'\Lambda_{y'}C_1 + \Lambda C_1 + \Lambda_{xy'} + \Psi_{y'y'}) + 2\Psi_{y'} + \Lambda_x \\ + y'\Lambda_{xy'} + y'\Psi_{y'y'} = 0.\end{aligned} \tag{2.83}$$

The problem now is that, due to the exponential on the r.h.s. of equation (2.82), differently from Case D, we are not able to obtain a second expression for $p'(x)$ by differentiating w.r.t. x. The alternative we have found can be summarized as follows. We first note that if $\Lambda_{y'} = 0$, equation (2.83) is already a linear algebraic equation† for p', so that we are only worried with the case $\Lambda_{y'} \neq 0$. With this in mind, we divide equation (2.83) by $\Lambda_{y'}$ and, *if* the resulting expression depends on y', we directly obtain a linear algebraic equation in $p'(x)$ by just differentiating w.r.t. y'.

Example.‡

$$y'' = \frac{y'(xy' + 1)(-2 + e^y)}{y'x^2 + y' - 1}. \tag{2.84}$$

We determine Υ, $w(x, y)$, and $\mathcal{H}(y + p(x))$ as

$$\Upsilon = \frac{y'(xy' + 1)e^y}{y'x^2 + y' - 1}; \qquad w(x, y) = e^y; \qquad \mathcal{H} = 1. \tag{2.85}$$

†We can see this by assuming that $\Lambda_{y'} = 0$ and that equation (2.83) does not contain p', and then arriving at a contradiction as follows. We first set the coefficients of p' in equation (2.83) to zero, arriving at

$$0 = C_1\Lambda + \Psi_{y'y'} = 2\Psi_{y'} + \Lambda_x + \Psi_{y'y'}y'. \tag{A}$$

Eliminating $\Psi_{y'y'}$ gives

$$2\Psi_{y'} = C_1\Lambda y' - \Lambda_x.$$

Differentiating the expression above w.r.t. y' and since $\Lambda_{y'} = 0$, we have

$$2\Psi_{y'y'} = C_1\Lambda.$$

Finally, using equation(A), $0 = \Lambda$, contradicting $F_{y'} \neq 0$.

‡There are no examples of this type in all of Kamke's set of non-linear second-order ODEs.

From the last equation we know that we are in Case E. We then determine Λ and Ψ as in equations (2.80) and (2.81):

$$\Lambda = \frac{y'x^2 + y' - 1}{y'(xy'+1)}$$
$$\Psi = -2. \tag{2.86}$$

Now, we build equation (2.82):

$$\frac{1}{xy'+1}\left(\left(p'' + p'^2 + y'^2\frac{(xp'-1)}{xy'+1}\right)\left(x^2 + 1 - \frac{1}{y'}\right) + 2xp' - 2\right) = \frac{C_3}{C_2 e^p} \tag{2.87}$$

and, differentiating w.r.t. y', we obtain equation (2.83):

$$\frac{2xy' + 1 - (x^3 + x)y'^2}{y'^2(xy'+1)^2}(p'' + p'^2) + \frac{2y' - 1 - 2x + xy'}{(xy'+1)^3}(xp' - 1) = 0. \tag{2.88}$$

Proceeding as explained, dividing by $\Lambda_{y'}$ and differentiating w.r.t. y', we have

$$\frac{\partial}{\partial y'}\left(y'^2 \frac{2y' - 1 - 2x + xy'}{(xy'+1)(2xy' + 1 - (x^3 + x)y'^2)}\right)(xp' - 1) = 0. \tag{2.89}$$

Solving for $p'(x)$ gives $p'(x) = 1/x$, from which equation (2.50) becomes:

$$\mathcal{F}(x, y') = \left(y' - \frac{1}{x}\right)\frac{y'x^2 + y' - 1}{y'(xy'+1)}. \tag{2.90}$$

Case F: The final branch occurs when equation (2.83) divided by $\Lambda_{y'}$ does not depend on y' (so that we will not be able to differentiate w.r.t. y'). In this case we can build a linear algebraic equation for $p'(x)$ as follows. Let us introduce the label $\beta(x, p', p'')$ for equation (2.83) divided by $\Lambda_{y'}$, so that equation (2.83) becomes:

$$\Lambda_{y'}(x, y')\beta(x, p', p'') = 0. \tag{2.91}$$

Since we obtained equation (2.83) by differentiating equation (2.82) w.r.t. y', equation (2.82) can be written in terms of β by integrating equation (2.91) w.r.t. y':

$$\Lambda(x, y')\beta(x, p', p'') + \gamma(x, p', p'') = \frac{C_3}{C_2 e^{p(x)C_1}} \tag{2.92}$$

where $\gamma(x, p', p'')$ is the constant of integration, and can be determined explicitly in terms of x, p' and p'' by comparing equation (2.92) with equation (2.82). Taking into account that $\beta(x, p', p'') = 0$, equation (2.92) reduces to:

$$\gamma(x, p', p'') = \frac{C_3}{C_2 e^{p(x)C_1}}. \tag{2.93}$$

We can remove the unknowns C_2 and C_3 after multiplying equation (2.93) by $e^{p(x)C_1}$, differentiating w.r.t. x, and then dividing once again by $e^{p(x)C_1}$. We now have our second equation for p', which we can build explicitly in terms of p', since we know $\gamma(x, p', p'')$ and C_1:

$$\frac{d\gamma}{dx} + C_1 p'\gamma = 0. \tag{2.94}$$

Eliminating the derivatives of p' between equations (2.91) and (2.94) leads to a linear algebraic equation in p'. Once we have p', the determination of $\mathcal{F}(x, y')$ follows directly from equation (2.50).

2.3. INTEGRATING FACTORS OF THE FORM $\mu(y, y')$

From equation (2.11), the ODE family admitting an integrating factor of the form $\mu(y, y')$ is given by

$$y'' = -\frac{y'}{\mu}\left(G_y + \frac{\partial}{\partial y}\int \mu dy'\right) - \frac{G_x}{\mu} \tag{2.95}$$

where $\mu(y, y')$ and $G(x, y)$ are arbitrary functions of their arguments. For this ODE family, it would be possible to develop an analysis and split the problem into cases as done in the previous section for the case $\mu(x, y')$. However, it is straightforward to note that under the transformation $y(x) \to x$, $x \to y(x)$, equation (2.95) transforms into an ODE of the form equation (2.23) with integrating factor $\mu(x, {y'}^{-1})/y'^2$. It follows that an integrating factor for any member of the ODE family above can be found by merely changing variables in the given ODE and calculating the corresponding integrating factor of the form $\mu(x, y')$.

EXAMPLE.

$$y'' - \frac{{y'}^2}{y} + \sin(x)y'y + \cos(x)y^2 = 0. \tag{2.96}$$

Changing variables $y(x) \to x$, $x \to y(x)$ we obtain

$$y'' + \frac{y'}{x} - \sin(y){y'}^2 x - \cos(y)x^2{y'}^3 = 0. \tag{2.97}$$

Using the algorithm outlined in the previous section, an integrating factor of the form $\mu(x, y')$ for equation (2.97) is given by

$$\frac{1}{{y'}^2 x} \tag{2.98}$$

from where an integrating factor of the form $\mu(y, y')$ for equation (2.96) is $1/y$, leading to the first integral

$$\sin(x)y + \frac{y'}{y} + C_1 = 0, \tag{2.99}$$

which is a first-order ODE of Bernoulli type. The solution to equation (2.96) then follows directly. This example is interesting as from González-López (1988), equation (2.96) has no point symmetries.

3. Integrating Factors and Symmetries

Besides the formulas for integrating factors of the form $\mu(x, y)$, the main result presented in this paper is a systematic algorithm for the determination of integrating factors of the form $\mu(x, y')$ and $\mu(y, y')$ *without solving any auxiliary differential equations or performing differential Gröbner basis calculations*, and these last two facts constitute the relevant point. Nonetheless, it is interesting to briefly compare the standard integrating factor (μ) and symmetry approaches, so as to have an insight of how complementary these methods can be in practice.

To start with, both methods tackle an *nth* order ODE by looking for solutions to a linear *nth order determining PDE* in $n+1$ variables. Any given ODE has infinitely many integrating factors and symmetries. When many solutions to these *determining* PDEs are found, both approaches can, in principle, give a multiple reduction of order.

In the case of integrating factors there is one unknown function, while for symmetries there is a pair of infinitesimals to be found. But symmetries are defined up to an arbitrary function, so that we can always take one of these infinitesimals equal to zero;[†] hence we are facing approaches of equivalent levels of difficulty and actually of equivalent solving power.

Also valid for both approaches is the fact that, unless some *restrictions* are introduced on the functional dependence of μ or the infinitesimals, there is no hope that the corresponding determining PDEs will be easier to solve than the original ODE. In the case of symmetries, it is usual to restrict the problem to ODEs having *point symmetries*, that is, to consider infinitesimals depending only on x and y. The restriction to the integrating factors here discussed is similar: we considered μ's depending on only two variables.

At this point it can be seen that the two approaches are complementary: the determining PDEs for μ and for the symmetries are different,[‡] so that even using identical restrictions on the functional dependence of μ and the infinitesimals, problems which may be untractable using one approach may be easy or even trivial using the other one.

As an example of this, consider Kamke's ODE 6.37

$$y'' + 2yy' + f(x)(y' + y^2) - g(x) = 0. \tag{3.1}$$

For *arbitrary* $f(x)$ and $g(x)$, this ODE has an integrating factor depending only on x, easily determined using the algorithms presented. Now, for *non-constant* $f(x)$ and $g(x)$, this ODE has no point symmetries, that is, no infinitesimals of the form $[\xi(x, y), \eta(x, y)]$, except for the particular case in which $g(x)$ can be expressed in terms of $f(x)$ as in[§]

$$g(x) = \frac{f''}{4} + \frac{3ff'}{8} + \frac{f^3}{16} - \frac{C_2 \exp\left(-3/2 \int f(x)\,dx\right)}{4\left(2C_1 + \int \exp\left(-1/2 \int f(x)\,dx\right) dx\right)^3}. \tag{3.2}$$

Furthermore, this ODE does not have non-trivial symmetries of the form $[\xi(x, y'), \eta(x, y')]$ either, and for symmetries of the form $[\xi(y, y'), \eta(y, y')]$, the determining PDE does not split into a system.

Another ODE example of this type is found in a paper by González-López (1988):

$$y'' - \frac{{y'}^2}{y} - g(x)py^p y' - g' y^{p+1} = 0. \tag{3.3}$$

In that work it is shown that for constant p, the ODE above, only has point symmetries for very restricted forms of $g(x)$. For instance, equation (2.96) is a particular case of

[†]Symmetries $[\xi(x, y, \ldots, y^{(n-1)}), \eta(x, y, \ldots, y^{(n-1)})]$ of an nth order ODE can always be rewritten as $[G, (G - \xi)y' + \eta]$, where $G(x, y, \ldots, y^{(n-1)})$ is an arbitrary function (for first-order ODEs, y' must be replaced by the r.h.s. of the ODE). Choosing $G = 0$, the symmetry acquires the form $[0, \bar{\eta}]$

[‡]We are considering here ODEs of order greater than one.

[§]To determine $g(x)$ in terms of $f(x)$ we used the *standard form* Maple package by Reid and Wittkopf (1993) complemented with some basic calculations.

the ODE above and has no point symmetries. On the other hand, *for arbitrary* $g(x)$, equation (3.3) has an obvious integrating factor depending on only one variable: $1/y$, leading to a first integral of Bernoulli type:

$$\frac{y'}{y} - g(x)y^p + C_1 = 0 \tag{3.4}$$

so that the whole family equation (3.3) is integrable by quadratures.

We note that equations (3.1) and (3.3) are, respectively, particular cases of the general reducible ODEs having integrating factors of the form $\mu(x)$:

$$y'' = -\frac{(\mu_x + G_y)}{\mu(x)}y' - \frac{G_x}{\mu(x)} \tag{3.5}$$

where $\mu(x)$ and $G(x, y)$ are arbitrary; and $\mu(y)$:

$$y'' = -\frac{(\mu_y y' + G_y)}{\mu(y)}y' - \frac{G_x}{\mu(y)}. \tag{3.6}$$

In turn, these are very simple cases if compared with the general ODE families equations (2.23) and (2.95), respectively having integrating factors of the forms $\mu(x, y')$ and $\mu(y, y')$, and which can be systematically reduced in order using the algorithms presented here.

It is then natural to conclude that the integrating factor and the symmetry approaches are useful for solving different types of ODEs, and can be viewed as equivalently powerful and general, and, in practice, complementary. Moreover, if for a given ODE, an integrating factor and a symmetry are known, in principle one can combine this information to build two first integrals and reduce the order by two at once (see for example Stephani (1989)).

4. Tests

After plugging the reducible-ODE scheme here presented into the ODEtools package (Cheb-Terrab *et al.*, 1998), we tested the scheme and routines using Kamke's non-linear 246 second-order ODE examples.[†] The purpose was to confirm the correctness of the returned results and to determine which of these ODEs have integrating factors of the form $\mu(x, y)$, $\mu(x, y')$ or $\mu(y, y')$. The test consisted of determining μ and testing the exactness condition equation (2.3).

In addition, we ran a comparison of performances in solving a subset of Kamke's examples having integrating factors of the forms $\mu(x, y')$ or $\mu(y, y')$, using different computer algebra ODE-solvers (Maple, Mathematica, MuPAD and the Reduce package Convode). The idea was to situate the new scheme in the framework of a sample of relevant packages presently available.

To run the comparison of performances, the first step was to classify Kamke's ODEs into: *missing x*, *missing y*, *exact* and *reducible*, where the latter refers to ODEs having integrating factors of the forms $\mu(x, y')$ or $\mu(y, y')$. ODEs missing variables were not included in the test since they can be seen as first-order ODEs in disguised form, and as such they are not the main target of the algorithm being presented. The classification we obtained for these 246 ODEs is as in Table 1.

[†]Kamke's ODEs 6.247 to 6.249 cannot be made explicit and are therefore excluded from the tests.

Table 1. Missing variables, exact and *reducible* Kamke's 246 second-order non-linear ODEs.

Classification	ODE numbers as in Kamke's book
99 ODEs are missing x or missing y	1, 2, 4, 7, 10, 12, 14, 17, 21, 22, 23, 24, 25, 26, 28, 30, 31, 32, 40, 42, 43, 45, 46, 47, 48, 49, 50, 54, 56, 60, 61, 62, 63, 64, 65, 67, 71, 72, 81, 89, 104, 107, 109, 110, 111, 113, 117, 118, 119, 120, 124, 125, 126, 127, 128, 130, 132, 137, 138, 140, 141, 143, 146, 150, 151, 153, 154, 155, 157, 158, 159, 160, 162, 163, 164, 165, 168, 188, 191, 192, 197, 200, 201, 202, 209, 210, 213, 214, 218, 220, 222, 223, 224, 232, 234, 236, 237, 243, 246
13 are in exact form	36, 42, 78, 107, 108, 109, 133, 169, 170, 178, 226, 231, 235
40 ODEs are *reducible* with integrating factor $\mu(x, y')$ or $\mu(y, y')$ *and* missing x or y	1, 2, 4, 7, 10, 12, 14, 17, 40, 42, 50, 56, 64, 65, 81, 89, 104, 107, 109, 110, 111, 125, 126, 137, 138, 150, 154, 155, 157, 164, 168, 188, 191, 192, 209, 210, 214, 218, 220, 222, 236
28 ODEs are *reducible* and not missing x or y	36, 37, 51, 66, 78, 97, 108, 123, 133, 134, 135, 136, 166, 169, 173, 174, 175, 176, 178, 179, 193, 196, 203, 204, 206, 215, 226, 235

Table 2. Performances in solving 28 Kamke's ODEs having an integrating factor $\mu(x, y')$ or $\mu(y, y')$.

	Kamke's ODE numbers			
	Convode	Mathematica 3.0	MuPAD 1.3	ODEtools
Solved:	51, 166, 173, 174, 175, 176, 179.	78, 97, 108, 166, 169, 173, 174, 175, 176, 178, 179, 206.	78, 97, 108, 133, 166, 169, 173, 174, 175, 176, 179,	51, 78, 97, 108, 133, 134, 135, 136, 166, 169, 173, 174, 175, 176, 178, 179, 193, 196, 203, 204, 206, 215.
Totals:	7	12	11	22
Reduced:				36, 37, 66, 123, 226, 235.
Totals:	0	0	0	6

For our purposes, the interesting subset is the one comprised of the 28 ODEs not already missing variables. The results we obtained using the aforementioned computer algebra ODE-solvers† are summarized as follows:‡

As shown above, while the scheme here presented is finding first integrals in all the 28 ODE examples, opening the way to solve 22 of them to the end, the next scores are only 12 and 11 ODEs, respectively solved by Mathematica 3.0 and MuPAD 1.3.

Concerning the six reductions of order returned by **odsolve**, it must be said that neither MuPAD nor Mathematica provide a way to convey them, so that perhaps their ODE-solvers are obtaining first integrals for these cases but the routines are giving up when they cannot solve the problem to the end.

†Maple R4 is not present in the table since it is not solving any of these 28 ODEs. This situation is being resolved in the upcoming Maple R5, where the ODEtools routines are included in the Maple library, and the previous ODE-solver was replaced by **odsolve**. However, the scheme here presented was not ready when the development library was closed; the *reducible* scheme implemented in Maple R5 is able to determine, when they exist, integrating factors only of the form $\mu(y')$.

‡Some of these 28 ODEs are given in Kamke (1959) in exact form and hence they can be easily reduced after performing a check for exactness; before running the tests all these ODEs were rewritten in explicit form by isolating y''.

5. Conclusions

In connection with second-order ODEs, this paper presented a systematic method for determining the existence of integrating factors and their explicit form, when they have the forms $\mu(x, y)$, $\mu(x, y')$ and $\mu(y, y')$. The scheme is new, as far as we know, and its implementation in the framework of the computer algebra package ODEtools has proven to be a valuable tool. Actually, the implementation of the scheme solves ODEs not solved by using standard or symmetry methods (see Section 3) or some other relevant and popular computer algebra ODE-solvers (see Section 4).

Furthermore, the algorithms presented involve only very simple operations and do not require solving auxiliary differential equations, except in one branch of the $\mu(x, y)$ problem. So, even for examples where other methods also work, for instance by solving the related PDE system equations (2.6) and (2.7) using ansätze and differential Gröbner basis techniques, the method here presented can return answers faster and avoiding potential explosions of memory.†

On the other hand, we have restricted the problem to the universe of second-order ODEs having integrating factors depending only on two variables while packages as CONLAW (in REDUCE) can try and in some cases solve the PDE system equations (2.6) and (2.7) by using more varied ansätze for μ.

A natural extension of this work would be to develop a scheme for building integrating factors of restricted but more general forms, now for higher order ODEs. We are presently working on these possible extensions,‡ and expect to succeed in obtaining reportable results in the near future.

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†Explosions of memory may happen when calculating all the integrability conditions involved at each step in the differential Gröbner basis approach.

‡See http://lie.uwaterloo.ca/odetools.html

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Appendix A

We display here both the integrating factors obtained for the 28 Kamke's ODEs used in the tests (see Section 4) and the "case" corresponding to each ODE when using just the algorithms for $\mu(x, y')$ or $\mu(y, y')$.† As explained in the proof of Lemma 3, the algorithm presented is subdivided into different cases: A, B, C, D, E and F, and case B is always either A or C.

Table 3. Integrating factors for Kamke's ODEs which are *reducible* and not missing x or y.

Integrating factor	Kamke's book ODE-number	Case
1	36	D
$e^{\int f(x)dx}$	37	A
y'^{-1}	51, 166, 169, 173, 175, 176, 179, 196, 203, 204, 206, 215	C
$\frac{b+y'}{(1+y'^2)^{3/2}}$	66	D
x	78	D
x^{-1}	97	A
y	108	D
y^{-1}	123	A
$\frac{1+y'}{(y'-1)y'}$	133	C
$\frac{y'-1}{(1+y')y'}$	134	C
$\frac{y'-1}{(1+y')(1+y'^2)}$	135	C
$\frac{y'-1}{h(y')}$	136	C
$\frac{x}{2xy'-1}$	174	C
$(1+y')^{-1}$	178	C
$\frac{1}{y'(1+2yy')}$	193	C
y'	226	A
$h(y')$	235	C

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†We note that for non-linear ODEs these two algorithms work as well when $\mu_{y'} = 0$, but in practice these very simple examples are covered by the algorithm for $\mu(x, y)$ presented in Section 2.1.