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SEVEN VARIATIONS on STANDARD BASES

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To MARINA
They know why
that know her

GENERAL INTRODUCTION

In recent years, originating both from Buchberger's concept of Gröbner bases [BUC1,2,3] for a polynomial ideal and his algorithm to compute them and from Hironaka's concept [HIR] of standard bases for a formal power series ideal, different instantiations of a general concept of Gröbner or standard bases have appeared, for ideals in polynomial rings, algebras of solvable types, graded rings, essentially with an aim to computational applications to commutative algebra, differential algebra, algebraic geometry, word problems.
A unifying approach to this concept has been however proposed only recently by Robbiano in the setting of valuation rings, by the introduction of the concept of a graded (or filtered) structure and of a general notion of standard basis in this context [ROB1] (similar unifying proposals can be found also in [SWE] and [B-S]).
Robbiano's notion, while able to encompass essentially all algebraic-oriented instantiations of a general concept of Gröbner or standard bases, is too wide to allow for a generalisation of Buchberger's algorithm (it is to be remarked that some of the concepts he generalised -for instance the original Hironaka notion- still lack an algorithm for their computation).
Originally motivated by the aim of finding suitable conditions on graded structures, under which the known algorithms (Buchberger's, the tangent cone algorithm [MOR1], Lazard's algorithm [LAZ]) could be generalized, I was gradually led toward a study of the basic properties of standard bases in graded structures. This paper presents the results of such investigation.
Working in such generality requires obviously some justification; I hope the following are sufficient. Any time one introduces a new instantiation of standard bases, one has to define more or less the same concepts, to state more or less the same results and, what is worse, to provide more or less the same proofs; I hope that the general concepts, results and proofs provided by this paper will help new researchers to avoid this trivial but cumbersome burden.
Also, some of the many computational applications of the original

Buchberger's concepts could hopefully be found to have a wider scope than polynomial rings.

This paper would not have been possible without the stimulating confrontation with many friends and colleagues working in the same area: the list of authors at the end of paper is only a subset of the people to which I am strongly indebted.

PART I
STANDARD BASES AND CAUCHY SEQUENCES

INTRODUCTION

A graded structure [ROB1] is a quintuple $\mathbf{A} := (A, \Gamma, w, G, F)$ where A is a ring, Γ is an ordered semigroup, $v: A-\{0\} \to \Gamma$ is a function which has the usual properties of a valuation, G is a Γ-graded ring, $in: A \to G$ a function which, to an element $a \in A$, associates a homogeneous element of G, which can be thought as an "initial form" of a.
In graded structures, a standard basis for an ideal I can be defined as a subset F of I, s.t. either:

A) each element $f \in I$ can be represented as $f = \sum g_i f_i$, with $g_i \in A$, $f_i \in F$ and $v(g_i) + v(f_i) \leq v(f)$

B) $\{in(f) : f \in F\}$ generates the homogeneous ideal $in(I) := (in(f) : f \in I-\{0\})$

The two conditions are equivalent if Γ is well-ordered (or more generally if I satisfies a "Krull-like" property); however, in general, as Robbiano showed ([ROB1], Th.3.1), while the first condition always implies the second one, the two conditions are not equivalent.
The first part of this paper deals with the problem of the equivalence of conditions A) and B).
Theorem 3.1 of [ROB1] attacks the problem stating a general condition -which is met in many important situations- under which the two conditions are equivalent.
I propose a different approach to the problem, namely to find conditions which are equivalent to B), and so, in general, weaker than A), which however are still related with representations of elements of I, which are "bounded" with respect to the valuation v.
One reason for this is that the known algorithms for standard basis computation are devised in order to compute a set satisfying condition B), but they work verifying the validity of condition A) for a finite subset of I. Therefore in order to generalize such algorithms, one needs to know an A)-like condition equivalent to B).
I will give a solution to this problem, with a limitation on the ordering on Γ, *inf-limitedness*, which, essentially, has the effect to allow to substitute a finitistic proof schema to the unconstructive Krull

property of [ROB1].

If A is commutative and noetherian, the equivalent condition is:

A') each element $f \in I A\hat{}$ can be represented as $f = \sum g_i f_i$, with $g_i \in A\hat{}$, $f_i \in F$ and $\upsilon(g_i) + \upsilon(f_i) \leq \upsilon(f)$

where $A\hat{}$ denotes the completion of A w.r.t. the topology induced by υ (obviously υ and in extend easily to $A\hat{}$).

However in general a characterization of standard sets similar to A') cannot be hoped for, and the best available condition in terms of "bounded representations" is the following:

A") each element of $I\hat{}$ can be obtained as a limit of a Cauchy sequence of elements of I, each of them having a representation in terms of F which is "bounded w.r.t. the valuation υ".

After recalling the notion of graded structures and illustrating it with several examples (§1), I introduce the notion of *standard set* (a set satisfying condition B) for an ideal) (§2) and an equivalent A)-like condition (§3), which is then interpreted in terms of ring completions (§4) and strengthened in the commutative-noetherian case (§5)

1 GRADED STRUCTURES

1.1 DEFINITION Let A be a ring with 1 (not necessarily a commutative or noetherian one); Γ a totally ordered semigroup, whose ordering is denoted by $<$; $\mathbf{F} := \{F(\gamma) : \gamma \in \Gamma\}$ a set of additive subgroups of A satisfying the following axioms:

(R1) $F(\gamma)$ is contained in $F(\delta)$ if $\gamma < \delta$

(R2) $F(\gamma)F(\delta)$ is contained in $F(\gamma + \delta)$

(R3) for each $a \in A$, $a \neq 0$, there is $\gamma \in \Gamma$ s.t. $a \in F(\gamma)$ and for each $\delta \in \Gamma$, s.t. $\delta < \gamma$, $a \notin F(\delta)$.

$\mathbf{A} := (A, \Gamma, \mathbf{F})$ is called a *filtered structure* ([ROB1]).

1.2 The following objects can be canonically associated to $\mathbf{A}$:

υ: $A - \{0\} \to \Gamma$, a function which to every $a \in A$, $a \neq 0$, associates the minimum $\gamma \in \Gamma$ s.t. $a \in F(\gamma)$;

$\mathbf{U} := \{U(\gamma) : \gamma \in \Gamma\}$, where, for each $\gamma \in \Gamma$, $U(\gamma) := \cup_{\delta<\gamma} F(\delta)$, an additive subgroup of A;

for each $\gamma \in \Gamma$, $G(\gamma) := F(\gamma)/U(\gamma)$, an additive group;

$G := \oplus_{\gamma\in\Gamma} G(\gamma)$, which is a Γ-graded ring, if endowed with the canonical multiplication;

in: $A \to G$, a function s.t. $in(0) = 0$ and, if $a \neq 0$, $in(a)$ is the residue class of a mod. $U(\upsilon(a))$.

(A,Γ,υ,G,in) is called the *graded structure* associated to $\mathbf{A}$ ([ROB1]), will also be denoted by $\mathbf{A}$, and it satisfies axioms A1-9 of [ROB1].

Also, the image of in is $\cup_{\gamma\in\Gamma} G(\gamma)$, so there are functions

$in^* : \cup_{\gamma \in \Gamma} G(\gamma) \to A$ s.t. in in* is the identity; we will fix such a function which moreover satisfies $in^*(1) = 1$.

1.3 Actually, the definition of filtered and graded structures in [ROB1] is restricted to the commutative case. However, the extension of the concepts and of the basic properties to non-commutative cases is straightforward.

1.4 EXAMPLE ([MAC]) Let $P := k[X_1,...X_n]$, k a field; let $\Gamma := \mathbb{N}$, with the usual ordering; for each $n \in \mathbb{N}$, let $F(n) := \{f \in P : deg(f) \le n\} \cup \{0\}$, let $\mathbf{F} := \{F(n) : n \in \mathbb{N}\}$. If $f \in P$, $f \neq 0$, then f can be uniquely written $f = \Sigma_{i=0...t} f_i$, f_i homogeneous, $deg(f_i) = i$, $f_t \neq 0$; define then $in(f) := f_t$; define $in(0) := 0$.
Then $(P,\mathbb{N},\mathbf{F})$ is a filtered structure; its associated graded structure is $(P,\mathbb{N},deg,P,in)$.

1.5 EXAMPLE More in general, let Γ be an ordered semigroup, and G a Γ-graded ring. Then each $g \in G$, $g \neq 0$ can be uniquely written as $g = \Sigma_{i=1...t} g_i$, g_i homogeneous, $g_i \neq 0$ and $deg(g_i) > deg(g_{i+1})$; define $deg(g) := deg(g_1)$, $in(g) := g_1$; define $in(0) := 0$. For each $\gamma \in \Gamma$, define $F(\gamma) := \{g \in G : deg(g) \le \gamma\} \cup \{0\}$.
Then $(G,\Gamma,\mathbf{F}) = (G,\Gamma,deg,G,in)$ is a filtered (graded) structure.

1.6 EXAMPLE ([BUC1],[MöM]) As a particular case, if $P := k[X_1,...,X_n]$, k a field, and $\mathbf{T}$ is the free (multiplicative) commutative semigroup generated by $\{X_1,...,X_n\}$, then P is a $\mathbf{T}$-graded ring: the set of homogeneous elements of degree $m \in \mathbf{T}$ is $\{c\, m : c \in k\}$.
If < is a total semigroup ordering on $\mathbf{T}$, then each $g \in P$, $g \neq 0$ can be uniquely written as $g = \Sigma_{i=1...t} c_i m_i$, $c_i \in k-\{0\}$, $m_i \in \mathbf{T}$, $m_i > m_{i+1}$; define $deg(g) := m_1$, $in(g) := c_1 m_1$; define $in(0) := 0$.
Then $(P,\mathbf{T},deg,P,in)$ is a graded structure.

1.7 EXAMPLE ([HIR]) Let $A := k[[X_1,...X_n]]$, $P := k[X_1,...X_n]$, k a field; if $f \in A$, $f \neq 0$, then f can be uniquely written $f = \Sigma_{i=t...\infty} f_i$, $f_i \in P$, f_i homogeneous, $deg(f_i) = i$, $f_t \neq 0$; the *order* of f is then defined to be $ord(f) := t$; define also $in(f) := f_t$; define $in(0) := 0$. For each $n \in \mathbb{N}$, let $F(n) := \{f \in P : ord(f) \ge n\} \cup \{0\}$.
If $\mathbb{N}$ is ordered by the inverse of the usual ordering, then $(A,\mathbb{N},\mathbf{F})$ is a filtered structure; its associated graded structure is $(A,\mathbb{N},ord,P,in)$.

1.8 EXAMPLE More in general, let A be a ring, $\mathbf{p} \subset A$ be an ideal, s.t.

$\cap_{\mathbb{N}} \mathbf{p}^n = 0$. Define $F(n) := \mathbf{p}^n$.

Then, if $\mathbb{N}$ is ordered as in 1.7, $(A,\mathbb{N},\mathbf{F})$ is a filtered structure.
If for each $f \in A$, $f \neq 0$, we define $v(f) := \min\{n \in \mathbb{N} : f \in \mathbf{p}^n\}$ and $in(f)$ as the residue class of f in $\mathbf{p}^n/\mathbf{p}^{n+1}$, where $n := v(f)$, defining also $in(0) := 0$, and G denotes the graded ring $\oplus_{\mathbb{N}} \mathbf{p}^n/\mathbf{p}^{n+1}$, then the associated graded structure is $(A,\mathbb{N},v,G,in)$.

1.9 EXAMPLE ([BER],[MOR2]) Let $\mathbf{S}$ be the free (multiplicative, non-commutative) semigroup generated by $\{X_1,...,X_n\}$, k a field, $P := k\langle \mathbf{S} \rangle$ be the non-commutative polynomial ring generated by $\{X_1,...,X_n\}$.
As in the commutative case, P is a $\mathbf{S}$-graded ring.
If $<$ is a total semigroup ordering on $\mathbf{S}$, then each $g \in P$, $g \neq 0$ can be uniquely written as $g = \Sigma_{i=1...t}\, c_i m_i$, $c_i \in k-\{0\}$, $m_i \in \mathbf{S}$, $m_i > m_{i+1}$; define $deg(g) := m_1$, $in(g) := c_1 m_1$; define $in(0) := 0$.
Then $(P,\mathbf{S},deg,P,in)$ is a graded structure.

1.10 EXAMPLE ([KRW]) A is called a *polynomial ring of solvable type* if there is a polynomial ring $P := k[X_1,...,X_n]$ and an ordering $<$ on its semigroup of terms $\mathbf{T}$ s.t., denoting $(P,\mathbf{T},deg_P,P,in_P)$ the graded structure defined in Ex. 1.6, then:

1) A coincides with P as a group.

2) The multiplication $\otimes$ in A is different from the one in P, but satisfies the following:

2.1) $c \otimes X_i = c\, X_i$, for all $c \in k$, for all i.

2.2) if $i \leq j$, $X_i \otimes X_j = X_i X_j$

2.3) if $i > j$, there is $h_{ij} \in P$ s.t. $X_i \otimes X_j = X_j X_i + h_{ij}$ and either $h_{ij} = 0$ or $deg_P(h_{ij}) < deg_P(X_j X_i)$.

For each $f \in A-\{0\}$, define $deg_R(f) := deg_P(f)$, $in_R(f) := in_P(f)$.
Then $(A,\mathbf{T},deg_R,P,in_R)$ is a graded structure.

2. STANDARD BASES AND STANDARD REPRESENTATIONS

2.1 Let $\mathbf{A} := (A,\Gamma,\mathbf{F}) = (A,\Gamma,v,G,in)$ be a filtered (graded) structure.
If $B \subset A-\{0\}$, let us denote $in\{B\} := \{in(b), b \in B\}$, $in(B)$ the two-sided ideal in G generated by $in\{B\}$.
If I is a two-sided ideal of A, and $B \subset I$ we say:

$f \in A-\{0\}$ has a *standard representation* in terms of B, iff it can be written as $f = \Sigma_{i=1...t}\, l_i b_i r_i$ with $l_i, r_i \in A-\{0\}$, $b_i \in B$, $v(l_i) + v(b_i) + v(r_i) \leq v(f)$;

$f \in A-\{0\}$ has a *truncated standard representation* at $\gamma \in \Gamma$ in terms of B, iff there is $g \in A$ s.t. $f = \Sigma_{i=1...t}\, l_i b_i r_i + g$ with $l_i, r_i \in A-\{0\}$,

$b_i \in B$, $v(l_i) + v(b_i) + v(r_i) \le v(f)$, either $g = 0$ or $v(g) < \gamma$;

$f \in A-\{0\}$ has a *Cauchy standard representation* in terms of B, iff for every $\gamma \in \Gamma$, f has a truncated standard representation at γ in terms of B.

B is a *standard set* for I in **A** iff in{B} generates in(I);

B is a *standard basis* for I in **A** if the following holds: $f \in I-\{0\}$ iff it has a standard representation in terms of B.

2.2 EXAMPLES In the following cases, B is a standard basis for I in **A** iff it is a standard set for I in **A**:

1) **A** is the graded structure of Ex. 1.4; in this case, the concept was introduced by Macaulay ([MAC]) and it is known as Macaulay basis or H-basis.

2) **A** is the graded structure of Ex.1.6, with **T** well-ordered by <; in this case, the concept was introduced by Buchberger ([BUC1], [BUC2], [BUC3]), together with an algorithm for its computation; it is known as Gröbner basis.

3) **A** is the graded structure of Ex.1.7, where it was introduced by Hironaka ([HIR]); this is the context in which the name of standard basis was introduced for the first time

4) **A** is either the graded structure of Ex.1.9, with **S** well-ordered by < or the graded structure of Ex.1.10, with **T** well-ordered by <; in both cases the concept is known as Gröbner basis.

5) **A** is any commutative noetherian graded structure satisfying the Krull property ([ROB1], Th.3.1)

2.3 EXAMPLES In general, however, while a standard basis is also a standard set, the converse is false, as the following examples show:

1) let $A := k[X]$, $\mathbf{p} := (X)$, and let $\mathbf{A} := (A,\mathbb{N},\mathrm{ord},G,\mathrm{in})$ be the graded structure defined in Ex.1.8; since $\mathbf{p}^n/\mathbf{p}^{n+1}$ is isomorphic to k, we can identify $G = \oplus_{\mathbb{N}} \mathbf{p}^n/\mathbf{p}^{n+1}$ with A.

Let $B := \{X-X^2\}$ so that $\mathrm{in}\{B\} = \{X\}$, $J := (B)$, $I := \mathbf{p}$, $f := X$.
Then, clearly B is a standard set and a standard basis for J in **A**, a standard set but not a standard basis for I in **A**, since $X \notin (X - X^2)$. However for each $n \in \mathbb{N}$, X has a truncated standard representation at n in terms of B, namely $X = (\Sigma_{i=0...n-1} X^i)\ (X-X^2) + X^{n+1}$, and therefore also a Cauchy standard representation.

2) let **S** be the the free non commutative semigroup generated by {X,Y}; if we assign $\deg(X) := 1$, $\deg(Y) := -1$, k<**S**> is a $\mathbb{Z}$-graded ring. Let $\mathbf{A} := (k\langle\mathbf{S}\rangle,\mathbb{Z},\deg,k\langle\mathbf{S}\rangle,\mathrm{in})$ be the graded structure defined in Ex. 1.5. Let $B := \{X-YXY\}$ so that $\mathrm{in}\{B\} = \{X\}$, $J := (B)$, $I := (X)$, $f := X$.
Then B is a standard set and a standard basis for J in **A**, a standard set but not a standard basis for I in **A**, since $X \notin (X - YXY)$.

However for each $n \in \mathbb{N}$, X has a truncated standard representation at $-2n$ in terms of B, namely $X = \Sigma_{i=0\ldots n} Y^i (X - YXY) Y^i + Y^{n+1} X Y^{n+1}$, and therefore also a Cauchy standard representation.

3) let $P := k[X_1, \ldots, X_n, \ldots]$ be the polynomial ring in infinitely many variables. If we assign $\deg(X_i) := -i$ for all i, P is a $\mathbb{Z}$-graded ring.
Let $\mathbf{A} := (P, \mathbb{Z}, \deg, P, in)$ be the graded structure defined in Ex. 1.5.
Let $B := \{X_i - X_{i+1} : i \geq 1\}$ so that $in\{B\} = \{X_i : i \geq 1\}$, $J := (B)$, $I := (X_i : i \geq 1)$, $f := X_1$.
Then B is a standard set and a standard basis for J in $\mathbf{A}$, a standard set but not a standard basis for I in $\mathbf{A}$, since $X_1 \notin (X_i - X_{i+1} : i \geq 1)$.
However for each $n \in \mathbb{N}$, X_1 has a truncated standard representation at $-n$ in terms of B, namely $X_1 = \Sigma_{i=1\ldots n} (X_i - X_{i+1}) + X_{n+1}$, and therefore also a Cauchy standard representation.

2.4 EXAMPLE In the case of Ex. 2.3.1) it is possible however to produce a standard representation of X in terms of B with coefficients in some extension ring of A, as follows:
Let A, $\mathbf{p}$, $\mathbf{A}$, B be as in Ex.2.3.1).
Let $A' := A_{\mathbf{p}}$, $\mathbf{p}' := \mathbf{p}A'$; let $\mathbf{A}' := (A', \mathbb{N}, ord', G', in')$ be the graded structure defined as in Ex.1.8; it is immediate that G' is isomorphic to A, and that the restrictions of ord' and in' to A coincide with ord and in.
Since $1 - X$ is an invertible element in $\mathbf{A}'$, $X = (1 - X)^{-1} (X - X^2)$ is a standard representation of X in terms of B in $\mathbf{A}'$.
In a similar way, let $A'' := k[[X]]$, $\mathbf{p}'' := \mathbf{p}A''$; let $\mathbf{A}'' := (A'', \mathbb{N}, ord'', G'', in'')$ be the graded structure defined as in Ex.1.8; it is immediate that G'' is isomorphic to A, and that the restrictions of ord'' and in'' to A coincide with ord and in.
We have that $X = (\Sigma_{i=0\ldots\infty} X^i) (X - X^2)$ is a standard representation of X in terms of B in $\mathbf{A}''$.
However, it is clear that in the cases of Ex.2.3.2),3), standard representations with coefficients in some extension ring cannot be found.

2.5 Let $\mathbf{A} := (A, \Gamma, \mathbf{F}) = (A, \Gamma, v, G, in)$ be a filtered (graded) structure, I a two-sided ideal of A, $B \subset I - \{0\}$, $f \in A - \{0\}$.
Let us recursively define three sequences of elements of A, $\mathbf{CZ}(f,B) := (f_n : n \in \mathbb{N})$, $\mathbf{CA}(f,B) := (g_n : n \in \mathbb{N})$, $\mathbf{CR}(f,B) := (h_n : n \in \mathbb{N})$ as follows (we will usually omit the dependence on B, if there is no ambiguity in the context):

i) $f_0 := f$, $g_0 := 0$

ii) if $f_i = 0$ define $h_i := 0$, $f_{i+1} := 0$, $g_{i+1} := g_i$

iii) if $f_i \neq 0$ and $in(f_i) \notin in(B)$, define $h_i := 0$, $f_{i+1} := f_i$, $g_{i+1} := g_i$

iv) otherwise $f_i \neq 0$ and $in(f_i) \in in(B)$. So there are $l_{ij}, r_{ij} \in G-\{0\}$, $b_{ij} \in B$, s.t. $in(f_i) = \Sigma\, l_{ij}\, in(b_{ij})\, r_{ij}$. Define then $h_i := \Sigma_j\, in^*(l_{ij})\, b_{ij}\, in^*(r_{ij})$, $f_{i+1} := f_i - h_i$, $g_{i+1} := g_i + h_i$.

Remark that **CZ**, **CA**, **CR** are not uniquely defined.

2.6 LEMMA With the notations of 2.5, the following hold:

i) for $i \geq 1$, if $f_{i-1} = 0$, then $f_i = 0$

ii) for $i \geq 1$, if $f_{i-1} \neq 0$ and $in(f_{i-1}) \notin in(B)$, then $f_i = f_{i-1}$

iii) for $i \geq 1$, if $f_{i-1} \neq 0$ and $in(f_{i-1}) \in in(B)$, then $v(h_i) = v(f_{i-1}) > v(f_i)$

iv) for each i, $g_i + f_i = f$

v) for each i, $g_i \in (B) \subset I$ and has a standard representation in terms of B

2.7 EXAMPLES 1) let **A**, B, I, f as in Ex.2.3.1).
If $f_n := X^{n+1}$, $g_n := X - X^{n+1}$ $h_n := X^n - X^{n+1}$, then $\mathbf{CZ}(f) = (f_n : n \in \mathbb{N})$ $\mathbf{CA}(f) = (g_n : n \in \mathbb{N})$, $\mathbf{CR}(f) = (h_n : n \in \mathbb{N})$; $g_n = (\Sigma_{i=0...n-1} X^i)\ (X-X^2)$ is a standard representation.

2) let **A**, B, I, f as in Ex.2.3.2).
If $f_n := Y^nXY^n$, $g_n := X - Y^nXY^n$, $h_n := Y^{n-1}XY^{n-1} - Y^nXY^n$, then $\mathbf{CZ}(f) = (f_n : n \in \mathbb{N})$ $\mathbf{CA}(f) = (g_n : n \in \mathbb{N})$, $\mathbf{CR}(f) = (h_n : n \in \mathbb{N})$;
$g_n = \Sigma_{i=0...n-1}\, Y^i (X - YXY) Y^i$ is a standard representation.

3) let **A**, B, I, f as in Ex.2.3.3)
If $f_n := X_{n+1}$, $g_n := X_1 - X_{n+1}$, $h_n := X_n - X_{n+1}$, then $\mathbf{CZ}(f) = (f_n : n \in \mathbb{N})$ $\mathbf{CA}(f) = (g_n : n \in \mathbb{N})$, $\mathbf{CR}(f) = (h_n : n \in \mathbb{N})$; $g_n = \Sigma_{i=1...n} (X_i - X_{i+1})$ is a standard representation.

3 GRADED STRUCTURES WITH INF-LIMITED GRADUATIONS

3.1 Let $\mathbf{A} := (A,\Gamma,\mathbf{F}) = (A,\Gamma,v,G,in)$ be a filtered (graded) structure, I a two-sided ideal of A, $B \subset I-\{0\}$.

3.2 DEFINITION We say Γ is *inf-limited* iff:
for each $\gamma \in \Gamma$, for each infinite decreasing sequence $\gamma_1 > \gamma_2 > ... > \gamma_n > ...$ in Γ, there is n s.t. $\gamma_n \leq \gamma$.

3.3 REMARK If Γ is inf-limited, then either it is well-ordered (and therefore, $\gamma \geq 0$ for each $\gamma \in \Gamma$) or there is an infinite decreasing sequence $\gamma_1 > \gamma_2 > ... > \gamma_n > ...$ in Γ.
In the first case if f has a Cauchy standard representation in terms of B, then f has a standard representation in terms of B, since the

latter coincides with a truncated standard representation at 0. In particular a standard set is a basis of I, a statement which in general is false.
In the second case, for each $\gamma_1, \gamma_r, \gamma \in \Gamma$, there is $\gamma' \in \Gamma$ s.t. $\gamma_1 + \gamma' + \gamma_r < \gamma$, since $(\gamma_1 + \gamma_n + \gamma_r : n \in \mathbb{N})$ is an infinite decreasing sequence.

3.4 PROPOSITION If Γ is inf-limited, then the following conditions are equivalent:

A1) B is a standard set for I in **A**

A2) each $f \in I-\{0\}$ has a Cauchy standard representation in terms of B

A3) for each $f \in A-\{0\}$:

i) either f has a Cauchy standard representation in terms of B

ii) or there is $g \in A-\{0\}$ s.t. $in(g) \notin in(I)$ and $f - g$ has a standard representation in terms of B

Proof: A2⇒A1: (cf. [ROB1] theorem 3.1). Let $m \in in(I)$, $f \in I$ be s.t. $m = in(f)$, $\gamma := deg(m)$; let $f = \Sigma\, l_i b_i r_i + g$ be a truncated standard representation at γ. Then $m = in(f) = \Sigma\; in(l_i) in(b_i) in(r_i)$, the sum being done over those indexes s.t. $v(l_i) + v(b_i) + v(r_i) = \gamma$.

A1⇒A3: let $f \in A-\{0\}$ and let $(f_n : n \in \mathbb{N}) := \mathbf{CZ}(f)$, $(g_n : n \in \mathbb{N}) := \mathbf{CA}(f)$.
If for some n, $f_{n+1} = 0$, then $f = g_n$ has a standard representation.
If for some n, $in(f_n) \notin in(B) = in(I)$, then $f - f_n = g_n$ has a standard representation.
Finally, if for each n, $f_n \neq 0$ and $in(f_n) \in in(B) = in(I)$, then let $\gamma_n := v(f_n)$; then $(\gamma_n : n \in \mathbb{N})$ is an infinite decreasing sequence in Γ.
For each $\gamma \in \Gamma$, there is then n s.t. $\gamma_n < \gamma$.
Then f has a truncated standard representation at γ, since $f = g_n + f_n$, g_n has a standard representation and $v(f_n) < \gamma$.

A3⇒A2: Let $f \in I-\{0\}$ be s.t. it doesn't have Cauchy standard representations in terms of B. Then, by A3, there is $g \in A-\{0\}$ s.t. $in(g) \notin in(I)$ and f-g has a standard representation in terms of B.
We get therefore a contradiction, since f and $f - g$ are in I, so that $g \in I$ and $in(g) \in in(I)$.

3.5 LEMMA Let $C(I) := \cap_\Gamma\, I + V(\gamma)$. Then C(I) is a two-sided ideal.
Proof: If Γ is well ordered, then $I = C(I)$ and there is nothing to prove.
Otherwise in Γ there is an infinite decreasing sequence

$\gamma_1 > \gamma_2 > ... > \gamma_n > ...$
In this case, let $f \in C(I)$, $l, r \in A-\{0\}$; we have to show that for each $\gamma \in \Gamma$, $l f r \in I + U(\gamma)$.
Let $\gamma_l := v(l)$, $\gamma_r := v(r)$. There is n s.t. $\gamma_l + \gamma_n + \gamma_r \leq \gamma$. Since $f \in I + U(\gamma_n)$, $f = g + h$, $g \in I$, $v(h) < \gamma_n$. Then $l f r = l g r + l h r$, with $l g r \in I$, $v(l h r) < \gamma_l + \gamma_n + \gamma_r \leq \gamma$, so $l f r \in I + U(\gamma)$.

3.6 PROPOSITION The following condition is equivalent to A1, A2 and A3:
A4) $f \in C(I)-\{0\}$ iff f has a Cauchy standard representation in terms of B.

Proof: A4$\Rightarrow$A2: Since $I \subset C(I)$, it is obvious.
A2$\Rightarrow$A4: Let $f \in C(I)-\{0\}$, $\gamma \in \Gamma$. Then $f = f' + g'$, with $f' \in I$, $v(g') < \gamma$. If $v(f) < \gamma$, w.l.o.g. we can assume $f' = 0$, and there is nothing to prove. Otherwise $v(f') = v(f)$. Then, by A2), $f' = \sum l_i b_i r_i + g$, with $v(g) < \gamma$ and $v(l_i) + v(b_i) + v(r_i) \leq v(f')$, so $f = \sum l_i b_i r_i + (g + g')$, with $v(f) = v(f') \geq v(l_i) + v(b_i) + v(r_i)$, $v(g + g') < \gamma$.
Conversely assume f is s.t. for each $\gamma \in \Gamma$, $f = \sum l_i b_i r_i + g$, with $l_i, r_i \in A-\{0\}$, $b_i \in B$, $v(l_i)+v(b_i)+v(r_i) \leq v(f)$, $v(g) < \gamma$. So $f \in I + U(\gamma)$ for each $\gamma \in \Gamma$ and $f \in C(I)$.

4 INF-LIMITED GRADUATIONS AND COMPLETIONS

4.1 If Γ is well-ordered, the topology induced by the filtration $\mathbf{U} = \{U(\gamma) : \gamma \in \Gamma\}$ is the discrete topology, so all the results of this paragraph will hold trivially. To avoid this trivial case we will assume, throughout the paragraph, that Γ is inf-limited but not well-ordered. In this case there exists an infinite decreasing sequence $\gamma_1 > \gamma_2 > ... > \gamma_n > ...$ of elements of Γ. We will fix such a sequence throughout this and the following paragraphs.

4.2 LEMMA If Γ is inf-limited, A is a topological ring w.r.t. the filtration $\mathbf{U} = \{U(\gamma) : \gamma \in \Gamma\}$.
Proof: We have to show that, for each $\gamma \in \Gamma$, there are $\gamma', \gamma'' \in \Gamma$ s.t. if $f \in U(\gamma')$, $g \in U(\gamma'')$, then $f g \in U(\gamma)$.
To prove this, fix an arbitrary $\gamma' \in \Gamma$, and let $\gamma'_1 > ... > \gamma'_n > ...$ the infinite decreasing sequence defined by $\gamma'_n := \gamma' + \gamma_n$. Then there is n s.t. $\gamma'_n < \gamma$. Defining $\gamma'' := \gamma_n$ one is through.

4.3 LEMMA Let A^ be the completion of A w.r.t. $\mathbf{U}$. Let $a \in A\hat{}-\{0\}$ and (a_n) be a Cauchy sequence in A converging to a. Then there is N s.t. if $n > N$, then $v(a_n) = v(a_N)$ and $in(a_n) = in(a_N)$

Proof: For each γ_n there is $d(n)$ s.t. if $p,q \geq d(n)$, then $a_p - a_q \in U(\gamma_n)$. If $a_{d(n)} \in U(\gamma_n)$, then for each $p > d(n)$, $a_p \in U(\gamma_n)$; therefore, if for each n $a_{d(n)} \in U(\gamma_n)$, then $a_n \to 0$.
So, there is n s.t. $v(a_{d(n)}) \geq \gamma_n$. Then for each $p > d_n$, $v(a_{d(n)}) = v(a_p)$ and $\mathrm{in}(a_p) = \mathrm{in}(a_{d(n)})$.

4.4 LEMMA If (a_n) and (b_n) are Cauchy sequences in A converging to $a \in A\hat{}-\{0\}$ then, for sufficiently large m,n, $\mathrm{in}(a_m) = \mathrm{in}(b_n)$.
Proof: $(a_n - b_n)$ converges to 0. So $v(a_m - b_m) < v(a_N)$ for large m and $\mathrm{in}(a_m) = \mathrm{in}(a_N) = \mathrm{in}(b_m)$ for large m.

4.5 COROLLARY Define $v\hat{}: A\hat{} \to \Gamma$, $\mathrm{in}\hat{}: A\hat{} \to G$ by $v\hat{}(a) := v(a_N)$, $\mathrm{in}\hat{}(a) := \mathrm{in}(a_N)$.
Then $\mathbf{A}\hat{} := (A\hat{},\Gamma,v\hat{},G,\mathrm{in}\hat{})$ is a graded structure; identifying A with its image in $A\hat{}$ under the canonical immersion $i: A \to A\hat{}$, v and in are the restrictions of $v\hat{}$ and $\mathrm{in}\hat{}$ to A.

4.6 LEMMA Let $I\hat{}$ denote the completion of I in $A\hat{}$.
Then $I\hat{}$ is a two-sided ideal and $C(I) = I\hat{} \cap A$.

4.7 THEOREM If Γ is inf-limited, then the following conditions are equivalent:

- A1) B is a standard set for I in **A**
- A2) each $f \in I-\{0\}$ has a Cauchy standard representation in terms of B
- A3) for each $f \in A-\{0\}$:
  - i) either f has a Cauchy standard representation in terms of B
  - ii) or there is $g \in A-\{0\}$ s.t. $\mathrm{in}(g) \notin \mathrm{in}(I)$ and $f - g$ has a standard representation in terms of B
- A4) $f \in C(I)-\{0\}$ iff f has a Cauchy standard representation in terms of B.
- A5) B is a standard set for $I\hat{}$ in **A^**
- A6) $f \in I\hat{}$ iff there is a Cauchy sequence (g_n) of elements of I converging to f, s.t. for each n, g_n has a standard representation in terms of B
- A7) for each $f \in A\hat{}-\{0\}$:
  - i) if $f \in I\hat{}$, there is a Cauchy sequence (g_n) of elements of I converging to f, s.t. for each n, g_n has a standard representation in terms of B
  - ii) if $f \notin I\hat{}$, there is $g \in A\hat{}-\{0\}$ s.t. $\mathrm{in}\hat{}(g) \notin \mathrm{in}\hat{}(I\hat{})$ and

$f - g \in I$ has a standard representation in terms of B

Proof: $A1 \Leftrightarrow A5$: it is obvious since $in(I) = in\hat{}(I\hat{})$.

$A5 \Rightarrow A7$: Let $f \in A\hat{} - \{0\}$, and let $(f_n : n \in \mathbb{N}) := \mathbf{CZ}(f)$, $(g_n : n \in \mathbb{N}) := \mathbf{CA}(f)$.

Remark that, by the recursive definition, each $g_n \in I$.

CASE 1: for some n, $f_n = 0$.

Then $f = g_n$ has a standard representation; therefore $f \in I$.

CASE 2: for some n, $f_n = f_{n+1} \neq 0$.

Let n be the least such index. Then $in\hat{}(f_n) \notin in\hat{}(I\hat{})$ and $f - f_n = g_n$ has a standard representation; since $f_n \notin I\hat{}$ and $g_n \in I$, $f \notin I\hat{}$.

CASE 3: for all n, $0 \neq f_n \neq f_{n+1}$.

Then $in\hat{}(f_n) \in in\hat{}(I\hat{})$ for all n.

If we denote $\gamma'_n := v\hat{}(f_n)$, $(\gamma'_n : n \in \mathbb{N})$ is a decreasing sequence, so for each γ there is $\gamma'_n \leq \gamma$.

Therefore $\mathbf{CZ}(f)$ is a Cauchy sequence converging to 0, and, since $g_n + f_n = f$ for all n, this implies that $\mathbf{CA}(f)$ is a Cauchy sequence converging to f.

Therefore $f \in I\hat{}$ and $\mathbf{CA}(f)$ is the Cauchy sequence required by the thesis.

$A7 \Rightarrow A6$ Trivial

$A6 \Rightarrow A5$: let $m \in in\hat{}(I\hat{})$ and let $f \in I\hat{}$ be s.t. $m = in\hat{}(f)$.

Let (f_n) be the Cauchy sequence whose existence is implied by A6.

If n is sufficiently large $m = in\hat{}(f) = in(f_n)$.

Let $f_n = \Sigma_j l_j b_j r_j$ be a standard representation. Then $m = in(f_n) = \Sigma' in(l_j) in(b_j) in(r_j)$ where the sum is taken on those j's s.t. $v(f_n) = v(l_j) + v(b_j) + v(r_j)$.

4.8 PROPOSITION Let $f_1, f_2 \in A\hat{} - I\hat{}$ be s.t. $f_1 - f_2 \in I\hat{}$.

Let $g_1, g_2 \in A\hat{}$ be s.t. $in\hat{}(g_1) \notin in(I)$, $in\hat{}(g_2) \notin in(I)$, $f_1 - g_1 \in I$, $f_2 - g_2 \in I$ (such elements exist because of 4.7, A7).

Then $v\hat{}(g_1) = v\hat{}(g_2)$ and $in\hat{}(g_1) - in\hat{}(g_2) \in in(I)$.

Proof: Since $g_1 - g_2 \in I\hat{}$, if (say) $v\hat{}(g_1) > v\hat{}(g_2)$ then $in\hat{}(g_1 - g_2) = in\hat{}(g_1) \notin in(I)$, a contradiction.

So $v\hat{}(g_1) = v\hat{}(g_2)$ and $in\hat{}(g_1) - in\hat{}(g_2) = in\hat{}(g_1 - g_2) \in in(I)$.

4.9 REMARK If B is a standard set for $I \subset A$ and $f \in A\hat{} - I\hat{}$, by 4.7 A7), there is $g \in A\hat{} - \{0\}$ s.t. $in\hat{}(g) \notin in(I)$ and $f - g$ has a standard representation in terms of B; g is not unique; however, by 4.8, $v\hat{}(g)$ and the residue class of $in\hat{}(g)$ mod $in(I)$ are uniquely determined and depend only on the residue class of f mod. $I^\wedge$.

4.10 EXAMPLES In the three examples of 2.3 and 2.7, we have that $J^\wedge = I^\wedge = I A^\wedge$, $f \in C(I)$ and (g_n) is the Cauchy sequence converging to f, whose elements have a standard representation in terms of B.

4.11 EXAMPLE The following example shows that in 4.7 the assumption that Γ be inf-limited is essential:
Let $P := k[X,Y]$; **T** the semigroup of terms of P, ordered in such a way that $X^i Y^j < X^k Y^l$ iff $i > k$ or $i = k$ and $j > l$, so that **T** is not inf-limited ($Y^n > Y^{n+1} > X$ for all n); let **P** := (P,**T**,deg,P,in) be the graded structure defined as in Ex. 1.6. Let $B := \{X, Y - Y^2\}$, $I := (B)$, $J := (X,Y)$; then B is a standard set both for I and J in **P** and is a standard basis for I in **P**.
However Y does not have Cauchy standard representations in terms of B; in fact let us choose $\gamma := X$, and assume by contradiction there are p_1, p_2, $g \in k[X,Y]$ s.t. $\deg(g) < X$ (which implies $g \in (X)$), and
$Y - g = p_1 (Y-Y^2) + p_2 X$ be a standard representation.
Then necessarily $\deg(p_1) = 1$ (meaning $p_1(0,0) \neq 0$) and if
$p_1(X,Y) =: q(Y) + X r(X,Y)$, we should have $Y - q(Y)(Y - Y^2) \in (X)$, which is obviously impossible.

5 LEFT IDEALS - COMMUTATIVE STRUCTURES

5.1 Let **A** := (A,Γ,**F**) = (A,Γ,v,G,in) be a filtered (graded) structure.
If I is a left ideal of A, and $B \subset I$, let us denote $in_L(B)$ the left ideal in G generated by in(B).
We can trivially modify the definitions of 2.1 in order to define left standard representations, left truncated standard representations at $\gamma \in \Gamma$, left Cauchy standard representations, left standard sets, left standard bases.
If A (and so G) is commutative, each ideal is a left ideal, so we can drop "left" from the definitions above.

5.2 LEMMA Let I be a left ideal in A and let $C(I) := \cap_\Gamma I + V(\gamma)$.
Then C(I) is a left ideal in A; $I^\wedge$ is a left ideal in $A^\wedge$, $C(I) = I^\wedge \cap A$.

5.3 THEOREM Let **A** := (A,Γ,v,G,in) be a graded structure, with Γ inf-limited, **A^** := (A^,Γ,v^,G,in^) be its completion, I be a left ideal of A, $I^\wedge$ its completion in $A^\wedge$, $B \subset I-\{0\}$.
The following conditions are equivalent:

L1) B is a left standard set for I in **A**

L2) each $f \in I-\{0\}$ has a left Cauchy standard representation in terms of B

L3) for each $f \in A-\{0\}$:

i) either f has a left Cauchy standard representation in terms of B

ii) or there is $g \in A-\{0\}$ s.t. $in(g) \notin in_L(I)$ and $f-g$ has a left standard representation in terms of B

L4) $f \in C(I)-\{0\}$ iff f has a left Cauchy standard representation in terms of B.

L5) B is a left standard set for $I^\wedge$ in **A^**

L6) $f \in I^\wedge$ iff there is a Cauchy sequence (g_n) of elements of I converging to f, s.t. for each n, g_n has a left standard representation in terms of B

L7) for each $f \in A^\wedge-\{0\}$:

i) if $f \in I^\wedge$, there is a Cauchy sequence (g_n) of elements of I converging to f, s.t. for each n, g_n has a left standard representation in terms of B

ii) if $f \notin I^\wedge$, there is $g \in A^\wedge-\{0\}$ s.t. $in^\wedge(g) \notin in^\wedge_L(I^\wedge)$ and $f-g \in I$ has a left standard representation in terms of B

If G is noetherian, the following condition is equivalent to L1-L7:

L8) B is a left standard basis for $I^\wedge$ in **A^**

<u>Proof</u>: the equivalence of L1-L7 requires an argument similar to the one in the proof of Theorem 4.7.

L5⇒L8: Since G is noetherian, one can extract from B a finite subset B' s.t. in(B') generates in(I), so we can assume $B := \{b_1,\dots,b_t\}$.
Let $f \in I^\wedge-\{0\}$; $(f_n : n \in \mathbb{N}) :=$ **CZ**(f), $(g_n : n \in \mathbb{N}) :=$ **CA**(f), $(h_n : n \in \mathbb{N}) :=$ **CR**(f).

Remark that, by definition, for each h_n there are $a_{in} \in A$ s.t.
$h_n = \Sigma_i a_{in} b_i \in A$ and $in(h_n) = \Sigma_j in(a_{in}) in(b_i)$

For each i=1...t define the sequence (c_{in}) as follows:

$c_{i0} := 0$

$c_{in} := c_{in-1} + a_{in}$

Then $g_n = \Sigma_i c_{in} b_i$ for each n.

For each i, (a_{in}) is a Cauchy sequence converging to 0, so (c_{in}) is a Cauchy sequence too. Let c_i be its limit; one has $f = \Sigma_i c_i b_i$ and $v^\wedge(f) \ge v^\wedge(c_i) + v(b_i)$ for all i.

L8⇒L6: Assume $f \in I^\wedge$ so $f = \Sigma_{i=1\dots t} a_i b_i$, $a_i \in A^\wedge-\{0\}$, $b_i \in B$, $v^\wedge(a_i) + v^\wedge(b_i) \le v^\wedge(f)$.

For each n, for each i, there is m s.t. $\gamma_m + v(b_i) < \gamma_n$.

Choose $a_{in} \in A$ s.t. $a_i - a_{in} \in U^\wedge(\gamma_m)$ and define $f_n := \Sigma_i a_{in} b_i$.

(f_n) is the required Cauchy sequence, since for each γ, if n is s.t. $\gamma_n < \gamma$, then for $p,q > n$, $f_p - f_q = \Sigma_i ((a_{ip} - a_i) - (a_{iq} - a_i)) b_i \in U(\gamma_{\gamma_n})$ and $f - f_p \in U\hat{}(\gamma_{\gamma_n})$.

5.4 EXAMPLES 1) let **A**, B, I, f as in Ex.2.3.1). Then $f = (\Sigma_{i=0...\infty} X^i)(X-X^2)$ is a left standard representation.

2) let A, **A**, B, I, J, f, as in Ex. 2.3.2) (resp. Ex. 2.3.3)). Then it is easy to verify that f doesn't have *any* representation in terms of B in A^; therefore while I=C(J), B is *not* a basis of I^ in A.

PART II
STANDARD BASES AND SYZYGIES

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INTRODUCTION

The basic idea of Buchberger's algorithm for Gröbner bases is that the existence of a suitable representation (a Cauchy standard one, in the terminology of this paper) is not to be tested for all the elements of I but just for a finite, "critical", subset.
A rough description of Buchberger's algorithm and of most of its generalizations, is then as follows:

> given a finite basis F of an ideal I, a finite subset S of I is computed and the existence of a Cauchy standard representation is tested for each element in S; if the test is successful, then F is a standard set; otherwise finitely many elements in I are produced, which are adjoined to F; the whole procedure is then repeated again.

There are two main interpretations of this idea; one is in terms of critical-pair-completion algorithms and had a strong influence in rewrite-rule theory. The second one is in terms of syzygies and more algebraic in nature; it was first detected by Spear [SPE] and underlied the generalization of Buchberger's algorithm to polynomials over rings, studied by Zacharias [ZAC] and Schaller [SCH]; the most thorough treatment of this interpretation for polynomial rings is contained in [MÖL].
Such an interpretation was extended by Robbiano ([ROB1], Theorem 3.6 and Corollary 3.7) for commutative, noetherian graded structures.
The second part of this paper is mainly devoted to extend this result to non-noetherian, non-commutative cases.
I'm therefore able to show that, in order to ensure effective computation of standard sets, one at least needs effectiveness of the rings and functions involved in the definition, and the ability to compute in G homogeneous syzygies of ideals and homogeneous representations of homogeneous elements. In fact, under such effectiveness conditions, one can straightforwardly generalize Buchberger's algorithm.
However such a procedure is not guaranteed to halt, unless G is noetherian and im(v) well-ordered: non-noetherianity implies non termination in all cases in which I is finitely generated and in(I) is not such; while, if im(v) is not well-ordered, testing the existence of Cauchy standard representations could require an infinite computation.

A completion technique, different from Buchberger's, was proposed by Kandri-Rody and Weispfenning for algebras of solvable type [KRW]; since it cannot be interpreted in terms of syzygies, I shortly discuss a general version of their technique, too.
I discuss a syzygietic equivalent condition of the property of being a standard set, firstly for the easier case of left ideals (§1); then, after having built up the necessary algebraic machinery (§2), I extend it to two-sided ideals (§3); some examples are then presented (§4).
§5 is then devoted to Kandri-Rody and Weispfenning technique, while §6 introduces the notion of effectiveness for graded structures and a (non-terminating) procedure directly generalizing Buchberger's algorithm.

1 SYZYGIES FOR LEFT IDEALS

1.1 Let $\mathbf{A} := (A,\Gamma,\mathbf{F}) = (A,\Gamma,v,G,in)$ be a filtered (graded) structure.
Let $B \subset A-\{0\}$.
Let G^B be the free left module over G with basis $\{e_b : b \in B\}$, graded by assigning $deg(e_b) := deg(in(b)) = v(b)$.
Let $S: G^B \to G$ be the homogeneous morphism defined by $S(\Sigma_B\, n_b\, e_b) := \Sigma_B\, n_b\, in(b)$.
We will denote $LSyz(in(B)) := Ker(S) = \{\Sigma_B\, n_b\, e_b : \Sigma_B\, n_b\, in(b) = 0\}$, which is a left submodule of G^B.

1.2 Let A^B be the free left module over A with basis $\{e_b : b \in B\}$; let $s: A^B \to A$ be the morphism defined by $s(\Sigma_B\, a_b e_b) := \Sigma_B\, a_b b$.
If $f := \Sigma_B\, a_b e_b \in A^B$, define $v'(f) := max\{v(a_b) + v(b) : b \in B\}$,
$in'(f) := \Sigma_{B'}\, in(a_b)\, e_b \in G^B$, where $B' := \{b \in B : v(a_b) + v(b) = v'(f)\}$.

1.3 LEMMA Let Γ be inf-limited; $B' \supset B$; $F \subset A^B$ be s.t.
i) $\{in'(f) : f \in F\}$ is a homogeneous basis of $LSyz(in(B))$
ii) for each $f \in F$, $s(f)$ has a left Cauchy standard representation in terms of B'.

Let $h \in A-\{0\}$, $\gamma \in \Gamma$, $\gamma < v(h)$, be s.t. $h = \Sigma_{i=1...t}\, d_i\, b(i) + h'$, with $b(i) \in B$, $v(h) < \gamma_1 := max\{v(d_i) + v(b(i))\}$, and either $h' = 0$ or $v(h') < \gamma$.
Then $h = \Sigma_{j=1...s}\, d'_j\, b(j) + h''$, with $b(j) \in B'$,
$v(h) \le \gamma_2 := max\{v(d'_j) + v(b(j))\} < \gamma_1$, and either $h'' = 0$ or $v(h'') < \gamma$.
Proof: Let $h = \Sigma_{i=1...t}\, d_i\, b(i) + h'$, with $b(i) \in B$,
$v(h) < \gamma_1 := max\{v(d_i) + v(b(i))\}$, and either $h' = 0$ or $v(h') < \gamma$ and let $H := \Sigma_{i=1...t}\, d_i\, e_{b(i)} \in A^B$.

If $J := \{i : v(d_i) + v(b(i)) = \gamma_1\}$, then $\Sigma_J \, in(d_i) \, in(b(i)) = 0$, so
$\Sigma_J \, in(d_i) \, e_{b(i)} \in LSyz(in(B))$.
So there are $n_1,\ldots,n_r \in G-\{0\}$, homogeneous, $f_1,\ldots,f_r \in F$, s.t.
$\Sigma_J \, in(d_i) \, e_{b(i)} = \Sigma_{k=1\ldots r} \, n_k \, in'(f_k)$.
For each k, let $c_k := in^*(n_k)$; let $d_{jk} \in A-\{0\}$, $b(j,k) \in B$ be s.t.
$f_k = \Sigma_j \, d_{jk} \, e_{b(j,k)}$.
Then $H - \Sigma_k \, c_k \, f_k = \Sigma_{i=1\ldots t} \, d_i \, e_{b(i)} - \Sigma_k \, c_k \, \Sigma_j \, d_{jk} \, e_{b(j,k)} =: \Sigma_{l=1\ldots u} \, d'_l \, e_{b(l)}$.
Since $in'(\Sigma_{i=1\ldots t} \, d_i \, e_{b(i)}) - \Sigma_k \, n_k \, \Sigma_j \, in(d_{jk}) \, e_{b(j,k)} = 0$, then $v(d'_l) + v(b(l)) < \gamma_1$.
One has $h - h' = s(H) = \Sigma_l \, d'_l \, b(l) + \Sigma_k \, c_k \, s(f_k)$
If Γ is well-ordered, substituting to each $s(f_k)$ its left standard representation in terms of B', one is through.
Otherwise, for each k there is $\gamma_k \in \Gamma$ s.t. $v(c_k) + \gamma_k < \gamma$.
Each $s(f_k)$ has a left truncated standard representation at γ_k in terms of B', $s(f_k) = \Sigma_i \, c_{ik} \, b(i,k) + h_k$, with either $h_k = 0$ or $v(h_k) < \gamma_k$.
Therefore $h = \Sigma_l \, d'_l \, b(l) + \Sigma_k \, c_k \, \Sigma_i \, c_{ik} \, b(i,k) + (h + \Sigma_k \, c_k \, h_k)$ is the required representation.

1.4 COROLLARY Under the same assumptions of I, Th.5.3, if moreover B is a basis of I, L1-L7 are equivalent also to:

L9) there is a set $F \subset A^B$ s.t.:

i) $\{in'(f) : f \in F\}$ is a homogeneous basis of $LSyz(in(B))$

ii) for each $f \in F$, $s(f)$ has a left Cauchy standard representation in terms of B

<u>Proof</u>: L2⇒L9: trivial

L9⇒L2: Every $h \in I-\{0\}$ can obviously be represented as $h = \Sigma_{i=1\ldots t} \, d_i \, b(i) + h'$, with $b(i) \in B$, $v(h) \le \gamma_1 := \max\{v(d_i) + v(b(i))\}$, and $h' = 0$.
For each $\gamma \in \Gamma$, $\gamma < v(h)$ (otherwise there is nothing to prove), if $h = \Sigma_{i=1\ldots t} \, d_i \, b(i) + h'$, with $b(i) \in B$, $v(h) < \gamma_1 := \max\{v(d_i) + v(b(i))\}$, and either $h' = 0$ or $v(h') < \gamma$, then, by Lemma 1.3, there is a different representation $h = \Sigma_{j=1\ldots s} \, d'_j \, b(j) + h''$, with $b(j) \in B$, $v(h) \le \gamma_2 := \max\{v(d'_j) + v(b(j))\} < \gamma_1$, and either $h'' = 0$ or $v(h'') < \gamma$.
Therefore, since Γ is inf-limited, in a finite number of steps we will obtain a representation $h = \Sigma_{k=1\ldots u} \, d'_k \, b(k) + h''$, with $b(k) \in B$, $v(h) = \max\{v(d'_k) + v(b(k))\}$, and either $h'' = 0$ or $v(h'') < \gamma$; i.e. a left truncated standard representation at γ of h in terms of B.

1.5 Assume G is noetherian. Let B a finite standard set for I and let $F \subset A^B$ satisfying the conditions of L9).

For each $f \in F$, since $s(f)$ has a left Cauchy standard representation in terms of B in $\mathbf{A}$, it has a left standard representation in $\mathbf{A}$^. Let $s(f) = \Sigma_B a_b b$ be such a representation and define
$\mathrm{Lift}(f) := f - \Sigma_B a_b e_b \in (A\hat{})^B$.
Let $s\hat{}: (A\hat{})^B \to A\hat{}$ be the morphism defined by $s\hat{}(e_b) := b$.

1.6 COROLLARY $\{\mathrm{Lift}(f) : f \in F\}$ is a basis of $\mathrm{Ker}(s\hat{})$, actually a standard basis of $\mathrm{Ker}(s\hat{})$ under the "canonical" $\mathbf{A}$^-graded module structure on $\mathrm{Ker}(s\hat{})$.
<u>Sketch of proof</u>: if $f_0 \in \mathrm{Ker}(s\hat{})$, then $\mathrm{in}\hat{}'(f_0) \in \mathrm{Ker}(S)$, so
$\mathrm{in}\hat{}'(f_0) = \Sigma_F l_{f0}\, \mathrm{in}'(f)$, with l_{f0} homogeneous, $f_1 := f_0 - \Sigma_F \mathrm{in}^*(l_{f0}) f \in \mathrm{Ker}(s\hat{})$ and either $f_1 = 0$ or $v\hat{}'(f_1) < v\hat{}'(f_0)$.
Repeating the argument (as in I.2.5) we get Cauchy sequences (f_n) in $\mathrm{Ker}(s\hat{})$, $(\mathrm{in}^*(l_{fn}))$ in A for $f \in F$, all of them converging to zero, and such that, for all n, $f_{n+1} := f_n - \Sigma_F \mathrm{in}^*(l_{fn}) f$. Denoting, for $f \in F$, $l_f \in A\hat{}$ the limit of the Cauchy sequence $(\Sigma_{i=0...n} \mathrm{in}^*(l_{fi}))$, one obtains $f_0 = \Sigma_F l_f f$, with $v\hat{}'(f_0) \geq v\hat{}(l_f) + v\hat{}'(f)$.

2 FREE BIMODULES

2.1 In order to define modules of syzygies in the case of two-sided ideals, we have first to review the concept of free bimodules. Since I was unable to find a reference for a characterization of free bimodules, I will derive one based on the concept of tensor product of abelian groups, for which cf. [FUC], Ch.XI.

2.2 Let U, V be two abelian groups. Let $X := X(U,V)$ be the free abelian group generated by $\{(u,v) : u \in U, v \in V\}$ and let $Y := Y(U,V)$ be the subgroup generated by $\{(u_1 + u_2, v) - (u_1, v) - (u_2, v) : u_1, u_2 \in U, v \in V\} \cup$
$\cup \{(u, v_1 + v_2) - (u, v_1) - (u, v_2) : u \in U, v_1, v_2 \in V\}$.
Let $U \otimes V$ be the quotient group X/Y and denote $u \otimes v$ the coset of (u,v) in $U \otimes V$.
Then:

1) for each $u, u_1, u_2 \in U$, $v, v_1, v_2 \in V$, $n \in \mathbb{Z}$:

$$(u_1 + u_2) \otimes v = u_1 \otimes v + u_2 \otimes v$$
$$u \otimes (v_1 + v_2) = u \otimes v_1 + u \otimes v_2$$
$$u \otimes 0 = 0 \otimes v = 0$$
$$-u \otimes v = -(u \otimes v) = u \otimes (-v)$$
$$nu \otimes v = n(u \otimes v) = u \otimes nv$$

2) if $U = \oplus_{\gamma \in \Gamma} U_\gamma$, $V = \oplus_{\delta \in \Delta} V_\delta$, then $U \otimes V = \oplus_{\Gamma \times \Delta} U_\gamma \otimes V_\delta$.

2.3 Let A be a ring, B a set of indices, $\{e_b : b \in B\}$ a set of symbols, one for each element of B. Denote $(A\otimes A)^{|B|}$ by $(A\otimes A)^B$ and for each $u,v \in A$, $c \in B$ denote $u\, e_c\, v := (a_b : b \in B) \in (A\otimes A)^B$, with $a_b := 0$ if $b \neq c$, $a_c := u\otimes v$.

Then clearly $(A\otimes A)^B$ is isomorphic to the abelian group generated by $\{u\, e_b\, v : u,v \in A, b \in B\}$ with relations

$\{(u_1 + u_2)\, e_b\, v - u_1\, e_b\, v - u_2\, e_b\, v : u_1, u_2, v \in A, b \in B\} \cup$

$\cup \{u\, e_b\, (v_1 + v_2) - u\, e_b\, v_1 - u\, e_b\, v_2 : u, v_1, v_2 \in A, b \in B\}$.

2.4 $(A\otimes A)^B$ is turned into an A-bimodule if we define:

$$a\,(\Sigma_B \Sigma_i\, u_{bi}\, e_b\, v_{bi}) := \Sigma_B \Sigma_i\, (a\, u_{bi})\, e_b\, v_{bi}$$

$$(\Sigma_B \Sigma_i\, u_{bi}\, e_b\, v_{bi})\, a := \Sigma_B \Sigma_i\, u_{bi}\, e_b\, (v_{bi}\, a)$$

Moreover $(A\otimes A)^B$ is the free A-bimodule over $\{e_b : b \in B\}$, i.e.:

for each A-bimodule M, for each application $p : \{e_b : b \in B\} \to M$, there is a unique morphism $P : (A\otimes A)^B \to M$ s.t. $P(e_b) = p(e_b)$ for each $b \in B$.

Such a morphism is given by $P(\Sigma_B \Sigma_i\, u_{bi}\, e_b\, v_{bi}) = \Sigma_B \Sigma_i\, u_{bi}\, p(e_b)\, v_{bi}$.

2.5 If G is Γ-graded, and, for each $b \in B$, $\gamma_b \in \Gamma$, $(G\otimes G)^B$ is turned into a Γ-graded G-bimodule with $\deg(e_b) := \gamma_b$ for each $b \in B$, as follows: for $b \in B$, $\gamma \in \Gamma$, let $I(b,\gamma) := \{(\gamma',\gamma'') \in \Gamma\times\Gamma : \gamma' + \gamma_b + \gamma'' = \gamma\}$; let $G(\gamma',\gamma'') := G(\gamma') \otimes G(\gamma'')$; let $G(b,\gamma) := \oplus_{I(b,\gamma)}\, G(\gamma',\gamma'') \subset G\otimes G$.

Let $(G\otimes G)^B(\gamma) := \oplus_B\, G(b,\gamma)$, i.e. the subgroup of $(G\otimes G)^B$ generated by $\{u\, e_b\, v : b \in B, u \in G(\gamma'), v \in G(\gamma''), \gamma' + \gamma_b + \gamma'' = \gamma\}$.

Then $(G\otimes G)^B = \oplus_\Gamma\, (G\otimes G)^B(\gamma)$.

3 SYZYGIES FOR TWO-SIDED IDEALS

3.1 Let $\mathbf{A} := (A,\Gamma,F) = (A,\Gamma,v,G,in)$ be a filtered (graded) structure.

Let $B \subset A - \{0\}$.

Let $(G\otimes G)^B$ be the free bimodule over G with basis $\{e_b : b \in B\}$, graded by assigning $\deg(e_b) := \deg(in(b)) = v(b)$.

Let $S : (G\otimes G)^B \to G$ be the homogeneous morphism defined by $S(\Sigma_B \Sigma_i\, l_{ib}\, e_b\, r_{ib}) := \Sigma_B \Sigma_i\, l_{ib}\, in(b)\, r_{ib}$.

We will denote $Syz(in(B)) := Ker(S) = \{\Sigma_B \Sigma_i\, l_{ib}\, e_b\, r_{ib} : \Sigma_B \Sigma_i\, l_{ib}\, in(b)\, r_{ib} = 0\}$, which is a submodule of $(G\otimes G)^B$.

Let $(A\otimes A)^B$ be the free bimodule over A with basis $\{e_b : b \in B\}$; let

$s:(A\otimes A)^B \to A$ be the morphism defined by $s(\Sigma_B \Sigma_i l_{ib} e_b r_{ib}) := \Sigma_B \Sigma_i l_{ib} b r_{ib}$.

3.2 LEMMA Let $\Sigma_i a_i \otimes b_i \in A\otimes A$ with $v(a_i) = \gamma'$, $v(b_i) = \gamma''$ for all i, and let $m_i := in(a_i)$, $n_i := in(b_i)$.

Assume that $\Sigma_i m_i \otimes n_i = 0$.

Then there are a'_j, b'_j, with $v(a'_j) \le \gamma'$, $v(b'_j) \le \gamma''$ s.t. for all j either $v(a'_j) < \gamma'$ or $v(b'_j) < \gamma''$ and $\Sigma_i a_i \otimes b_i = \Sigma_j a'_j \otimes b'_j$.

<u>Proof</u>: The assumption implies that in $X(G(\gamma'),G(\gamma''))$:

(*) $\quad \Sigma_i (m_i, n_i) = \Sigma_k z_k ((u_{1k}, v_{1k}) - (u_{2k}, v_{2k}) - (u_{3k}, v_{3k}))$

where, for each k, $z_k \in \mathbb{Z}$, $u_{jk} \in G(\gamma')$, $v_{jk} \in G(\gamma'')$ and either $u_{1k} = u_{2k} + u_{3k}$, $v_{1k} = v_{2k} = v_{3k}$, or $u_{1k} = u_{2k} = u_{3k}$, $v_{1k} = v_{2k} + v_{3k}$.

For each j,k let $a_{jk} := in^*(u_{jk})$, $b_{jk} := in^*(v_{jk})$.

So, if $u_{1k} = u_{2k} + u_{3k}$, then either $a'_k := a_{1k} - a_{2k} - a_{3k} = 0$ or $v(a'_k) < \gamma'$;
and, if $v_{1k} = v_{2k} + v_{3k}$, then either $b'_k := b_{1k} - b_{2k} - b_{3k} = 0$ or $v(b'_k) < \gamma''$.

For each k, if $u_{1k} = u_{2k} = u_{3k}$, define $a'_k := 0$; if $v_{1k} = v_{2k} = v_{3k}$, define $b'_k := 0$.

Then $\Sigma_k z_k ((a_{1k} + a'_k),(b_{1k} + b'_k)) - (a_{2k}, b_{2k}) - (a_{3k}, b_{3k})) \in Y(A,A)$.

Since $X(G(\gamma'),G(\gamma''))$ is free, (*) implies that for each i there are k and j s.t. $(m_i,n_i) = (u_{jk},v_{jk})$.

If we define $(a''_i,b''_i) := (a_{jk},b_{jk})$, then, since $X(A,A)$ is free, (*) implies also:

$$\Sigma_i (a''_i,b''_i) = \Sigma_k z_k ((a_{1k},b_{1k}) - (a_{2k},b_{2k}) - (a_{3k},b_{3k}))$$

and therefore:

$$\Sigma_i a''_i \otimes b''_i = \Sigma_k z_k ((a_{1k} + a'_k) \otimes (b_{1k} + b'_k) - a_{2k} \otimes b_{2k} - a_{3k} \otimes b_{3k}) - \\ - \Sigma_k z_k a'_k \otimes b_{1k} - \Sigma_k z_k a_{1k} \otimes b'_k - \Sigma_k z_k a'_k \otimes b'_k$$

where the first sum is zero.

Also $a_i \otimes b_i = (a_i - a''_i) \otimes (b_i - b''_i) + a''_i \otimes (b_i - b''_i) + (a_i - a''_i) \otimes b_i + a''_i \otimes b''_i$.

Therefore:

$$\Sigma_i a_i \otimes b_i = \Sigma_i (a_i - a''_i) \otimes (b_i - b''_i) + \Sigma_i a''_i \otimes (b_i - b''_i) + \Sigma_i (a_i - a''_i) \otimes b''_i - \\ - \Sigma_k z_k a'_k \otimes b_{1k} - \Sigma_k z_k a_{1k} \otimes b'_k - \Sigma_k z_k a'_k \otimes b'_k$$

Since $in(a_i) = m_i = in(a''_i)$ and $in(b_i) = n_i = in(b''_i)$, one has $v(a_i - a''_i) < \gamma'$ and $v(b_i - b''_i) < \gamma''$, as required.

3.3 Let $f := \Sigma_{i=1\dots t} l_i e_{b(i)} r_i \in (A\otimes A)^B$, let
$\gamma := \max\{v(l_i) + v(b(i)) + v(r_i): i=1\dots t\}$, $g := \Sigma_{i=1\dots t} l'_i e_{b(i)} r'_i \in (G\otimes G)^B$, where
$l'_i := in(l_i)$ and $r'_i := in(r_i)$ iff $v(l_i) + v(b(i)) + v(r_i) = \gamma$,

$l'_i := 0$ and $r'_i := 0$ otherwise.

Remark that, if $g \neq 0$, then g and γ don't depend on the particular representation of f we are given.

Therefore, if $g \neq 0$, define $v'(f) := \gamma$, $in'(f) := g$.

3.4 LEMMA Let $f = \Sigma_i l_i e_{b(i)} r_i \in (A \otimes A)^B$, γ, g be as in 3.3.

If $g = 0$, then there are $l'_j, r'_j \in A$, $b'(j) \in B$ s.t.

i) $\Sigma_i l_i e_{b(i)} r_i = \Sigma_j l'_j e_{b'(j)} r'_j$

ii) $\gamma' := \max\{v(l'_j) + v(b'(j)) + v(r'_j)\} < \gamma$.

Proof: W.l.o.g. we can assume that for each i, $v(l_i) + v(b(i)) + v(r_i) = \gamma$.

Let $I(b) := \{i : b(i) = b\}$; for each γ', γ'' s.t. $\gamma' + v(b) + \gamma'' = \gamma$, let $I(b, \gamma', \gamma'') := \{i : b(i) = b,\ v(l_i) = \gamma',\ v(r_i) = \gamma''\}$.

Because of the direct sum decompositions (cf.2.5) $(G \otimes G)^B(\gamma) = \oplus_B G(b, \gamma)$ and $G(b, \gamma) = \oplus_{I(b,\gamma)} G(\gamma', \gamma'')$, $\Sigma_i in(l_i) e_{b(i)} in(r_i) = 0$ implies $\Sigma_{I(b)} in(l_i) e_{b(i)} in(r_i) = 0$ for each b, and also $\Sigma_{I(b,\gamma',\gamma'')} in(l_i) e_{b(i)} in(r_i) = 0$.

The thesis follows then from 3.2.

3.5 Let $f = \Sigma_i m_i e_{b(i)} n_i$ be a non-zero homogeneous element of $(G \otimes G)^B$.

Let $c_i := in^*(m_i)$, $d_i := in^*(n_i)$.

Denote $in^*(f) := \Sigma_i c_i e_{b(i)} d_i$ and remark that $v'(in^*(f)) = deg(f)$ and $in'(in^*(f)) = f$.

3.6 LEMMA Let Γ be inf-limited; $B' \supset B$; $F \subset (G \otimes G)^B$ be a homogeneous basis of $Syz(in(B))$, s.t. for each $f \in F$, $s(in^*(f))$ has a Cauchy standard representation in terms of B'.

Let $h \in A - \{0\}$, $\gamma \in \Gamma$, $\gamma < v(h)$, be s.t. $h = \Sigma_{i=1...t} l_i b(i) r_i + h'$, with $b(i) \in B$, $v(h) < \gamma_1 := \max\{v(l_i) + v(b(i)) + v(r_i)\}$, and either $h' = 0$ or $v(h') < \gamma$.

Then $h = \Sigma_{j=1...s} l'_j b(j) r'_j + h''$, with $b(j) \in B'$, $v(h) \leq \gamma_2 := \max\{v(l'_j) + v(b(j)) + v(r'_j)\} < \gamma_1$, and either $h'' = 0$ or $v(h'') < \gamma$.

Proof: If $h = \Sigma_{i=1...t} l_i b(i) r_i + h'$, with $b(i) \in B$, $v(h) < \gamma_1 := \max\{v(l_i) + v(b(i)) + v(r_i)\}$, and either $h' = 0$ or $v(h') < \gamma$, let $H := \Sigma_{i=1...t} l_i e_{b(i)} r_i \in (A \otimes A)^B$, so $s(H) = h - h'$, $g := \Sigma_{i=1...t} m_i e_{b(i)} n_i \in (G \otimes G)^B$, where (as in 3.3)

$m_i := in(l_i)$ and $n_i := in(r_i)$ iff $v(l_i) + v(b(i)) + v(r_i) = \gamma$,

$m_i := 0$ and $n_i := 0$ otherwise.

CASE 1) $g = 0$

Then by Lemma 3.4, $H = \Sigma_{j=1...t'} l'_j e_{b(j)} r'_j$, with $b(j) \in B$,

$\gamma_2 := \max\{\upsilon(l_i) + \upsilon(b(i)) + \upsilon(r_i)\} < \gamma_1$.

So $h = \Sigma_{j=1...t'} l'_j\, b(j)\, r'_j + h'$, with $b(j) \in B$,

$\upsilon(h) \le \gamma_2 := \max\{\upsilon(l'_j) + \upsilon(b(j)) + \upsilon(r'_j)\} < \gamma_1$, and either $h' = 0$ or $\upsilon(h') < \gamma$.

CASE 2) $g \ne 0$, so $g = in'(H)$

Then $S(in'(H)) = 0$, $in'(H) \in Syz(in(B))$.

So there are $m_k, n_k \in G-\{0\}$, homogeneous, $f_k \in F$ s.t. $in'(H) = \Sigma_k m_k f_k n_k$.

Let $c_k := in^*(m_k)$, $d_k := in^*(n_k)$; let $c_{ik}, d_{ik}, b(i,k) \in B$ be s.t.

$in^*(f_k) := \Sigma_i c_{ik}\, e_{b(i,k)}\, d_{ik}$

Let $H_1 := H - \Sigma_k c_k\, in^*(f_k)\, d_k = \Sigma_{i=1...t} l_i\, e_{b(i)}\, r_i - \Sigma_k c_k \Sigma_i c_{ik}\, e_{b(i,k)}\, d_{ik}\, d_k$.

Since $in'(H) - \Sigma_k m_k \Sigma_i in(c_{ik})\, e_{b(i,k)}\, in(d_{ik})\, n_k = 0$, then we can apply the same argument as in case 1 to produce a representation

$H_1 = \Sigma_j c'_j\, e_{b(j)}\, d'_j$ with $\max\{\upsilon(c'_j) + \upsilon(b(j)) + \upsilon(d'_j)\} < \gamma_1$.

One has $h - h' = s(H) = s(H_1) + \Sigma_k c_k\, s(in^*(f_k))\, d_k =$

$= \Sigma_j c'_j\, b(j)\, d'_j + \Sigma_k c_k\, s(in^*(f_k))\, d_k$

If Γ is well-ordered, substituting to each $s(in^*(f_k))$ its standard representation in terms of B', one is through.

Otherwise, for each k there is $\gamma_k \in \Gamma$ s.t. $\upsilon(c_k) + \gamma_k + \upsilon(d_k) < \gamma$.

Each $s(in^*(f_k))$ has a truncated standard representation at γ_k in terms of B', $s(in^*(f_k)) = \Sigma_i l_{ik}\, b(i,k)\, r_{ik} + h_k$, with either $h_k = 0$ or $\upsilon(h_k) < \gamma_k$.

Therefore $h = \Sigma_j c'_j\, b(j)\, d'_j + \Sigma_k c_k \Sigma_i l_{ik}\, b(i,k)\, r_{ik}\, d_k + (h + \Sigma_k c_k\, h_k\, d_k)$ is the required representation.

3.7 COROLLARY Under the same assumptions of I, Th.4.7, if moreover B is a basis of I, A1-A7 are equivalent also to:

A9) if F is a homogeneous basis of Syz(in(B)), for each $f \in F$, $s(in^*(f))$ has a Cauchy standard representation in terms of B.

<u>Proof</u>: similar to the proof of Corollary 1.4

4 EXAMPLES

4.1 To give a concrete illustration of the results above, we give now an explicit representation of LSyz(in(B)) and Syz(in(B)) in case G is either the **T**-graded ring $k[X_1,...,X_n]$ or the **S**-graded ring $k\langle \mathbf{S}\rangle$, **T** and **S** as in I,1.6, I,1.9.

With some abuse of notations, we will identify **T**, resp. **S**, with its image in $k[X_1,...,X_n]$, resp. $k\langle \mathbf{S}\rangle$, and G^B with $\{\Sigma_B\, g_b\, e_b \mid : g_b \in G\} \subset (G \otimes G)^B$.

The following results are then easy to prove:

4.2 Let H be a finite subset of $\mathbf{T}$, $H' := \{\{m,n\} : m, n \in H, m \neq n\}$.
If $\{m,n\} \in H'$, let $t := \text{l.c.m.}(m,n)$, m', n' be s.t. $m'm = t = n'n$ and let $\text{lsyz}(m,n) := m' e_m - n' e_n$.
If $m \in H$, let $\text{Syz}(m) := \{x_i e_m - e_m x_i : i = 1 \ldots n\}$.
Then $\text{LSyz}(H)$ is generated by $\{\text{lsyz}(m,n) : \{m,n\} \in H'\}$ (so that we get the "critical pairs" of the original Buchberger algorithm); $\text{Syz}(H)$ is generated by $\{\text{lsyz}(m,n) : \{m,n\} \in H'\} \cup (\cup_{m \in H} \text{Syz}(m))$

4.3 Let H be a finite subset of $\mathbf{S}$.
Let $H'' := \{(m,n) : m,n \in H,\ m \neq n, \text{ there is } w_{mn} \in \mathbf{S} \text{ s.t. } m = w_{mn}\, n\}$.
If $(m,n) \in H''$, let $\text{lsyz}(m,n) := e_m - w_{mn}\, e_n$.
Then $\text{LSyz}(H)$ is generated by $\{\text{lsyz}(m,n) : (m,n) \in H''\}$

4.4 We recall [MOR2] that if $(m_1,m_2) \in \mathbf{S}^2$, the set of matches of (m_1,m_2), $M(m_1,m_2)$, is the finite set of all 4-tuples $(l_1,l_2,r_1,r_2) \in \mathbf{S}^4$ s.t. either:

1) $l_1 = r_1 = 1,\ m_1 = l_2\, m_2\, r_2$
2) $l_2 = r_2 = 1,\ m_2 = l_1\, m_1\, r_1$
3) $l_1 = r_2 = 1,\ l_2 \neq 1,\ r_1 \neq 1$, there is $w \in \mathbf{S}$, $m_1 = l_2\, w,\ m_2 = w\, r_1$
4) $l_2 = r_1 = 1,\ l_1 \neq 1,\ r_2 \neq 1$, there is $w \in \mathbf{S}$, $m_1 = w\, r_2,\ m_2 = l_1\, w$.

4.5 Let H be a finite subset of $\mathbf{S}$, $H' := \{\{m,n\} : m, n \in H, m \neq n\}$.
If $m,n \in H$, let $\text{Syz}(m,n) := \{l_m\, e_m\, r_m - l_n\, e_n\, r_n : (l_m,l_n,r_m,r_n) \in M(m,n)\}$.
Then $\text{Syz}(H)$ is generated by $(\cup_{\{m,n\} \in H'} \text{Syz}(m,n)) \cup (\cup_{m \in H} \text{Syz}(m,m))$.
We have therefore a syzygietic interpretation of the results and algorithms in [MOR2].

5 KANDRI-RODY - WEISSPFENNING RIGHT CLOSURE TECHNIQUE

5.1 Let $\mathbf{A} := (A,\Gamma,\mathbf{F}) = (A,\Gamma,v,G,\text{in})$ be a filtered (graded) structure, with Γ inf-limited.
Let $B \subset A-\{0\}$.
Denote by $I(B)$ (resp. $I_L(B)$, $I_R(B)$) the two-sided (resp. left, right) ideal generated by B.
Let $C := \{g \in G : g\, g' = g'\, g, \text{ for each } g' \in G\}$.
Then C is a commutative Γ-graded sub-ring of G, so G is a right C-algebra, generated by homogeneous elements.
Denote by $\text{Gen}(G,C)$ a set of homogeneous [illegible] of G as a right C-alg[illegible]

5.2 LEMMA (Kandri-Rody, Weispfenning) The following conditions are equivalent:

1) $in(B) = in_L(B)$

2) For each $x \in Gen(G,C)$, for each $b \in B$, $in(b)\,x \in in_L(B)$

3) For each $r \in G$ s.t. $r = x_1 ... x_t$, $x_i \in Gen(G,C)$, for each $b \in B$, $in(b)\,r \in in_L(B)$

<u>Proof</u>: 1)⇒2) is obvious.

2)⇒3): by trivial induction on t

3)⇒1): let $m \in in(B)$, then $m = \Sigma_i l_i\, in(b_i)\, r_i$, where w.l.o.g. each r_i is a word in Gen(G,C); since for each i, $in(b_i)\, r_i \in in_L(B)$, the thesis is obvious.

5.3 PROPOSITION (Kandri-Rody, Weispfenning) If B is a left standard set for $I_L(B)$, the following conditions are equivalent:

1) $in(B) = in_L(B)$

2) $I(B)^\wedge = I_L(B)^\wedge$

3) for each $x \in Gen(G,C)$, for each $b \in B$, $b\, in^*(x)$ has a left Cauchy standard representation in terms of B.

<u>Proof</u>: 1)⇒2) Let $f \in I(B)^\wedge$. Applying a "left" version of I.2.5 and I.2.6, if $(g_n : n \in \mathbb{N}) := \mathbf{CA}(f)$, $(f_n : n \in \mathbb{N}) := \mathbf{CZ}(f)$, we can w.l.o.g. assume $g_n \in I_L(B)$ for each n.

Then, since $in(B) = in_L(B) = in_L(I_L(B))$, we have also, for each n, either $f_n = 0$ or $f_n \neq f_{n+1}$. Therefore $f_n \to 0$ and $g_n \to f$, implying that $f \in I_L(B)^\wedge$.

2)⇒3) $b\, in^*(x) \in I(B) \subset I(B)^\wedge = I_L(B)^\wedge$; so the thesis is obvious.

3)⇒1) Let $b\, in^*(x) = \Sigma_i l_i b_i + c$ be a truncated left standard representation at $\gamma := v(b) + deg(x)$; then $in(b)\, x = \Sigma'_i\, in(l_i)\, in(b_i)$, where the sum is done on those i's s.t. $v(l_i) + v(b_i) = \gamma$.

The thesis then follows from Lemma 5.2.

5.4 COROLLARY If B is a left standard set of $I_L(B)$ and the conditions of 5.3 hold, then B is a standard set of I(B).

<u>Proof</u>: Because of the assumptions, one has:

$$in(B) = in_L(B) = in_L(I_L(B)) = in^\wedge_L(I_L(B)^\wedge) = in^\wedge(I(B)^\wedge) = in(I(B))$$

5.5 REMARK The converse can be false: let **S** be the free (multiplicative, non-commutative) semigroup generated by {X,Y}, (P,**S**,deg,P,in) the graded structure defined in Ex. I,1.9, B := {Y}. Then B is a standard set for I(B) and a left standard set for $I_L(B)$; however $in_L(B) \subsetneq in(B)$, since Y X is in the second but not in the first

ideal. A minimal left standard set of $I(B)$ is $B_1 := \{YX^n : n \in \mathbb{N}\}$, which obviously satisfies the assumptions of 5.3.
Remark also that $in_R(B_1) \subsetneq in(B_1) = in_L(B_1)$ so that, in this general situation, the left-right simmetry of Kandri-Rody and Weispfenning result is not preserved.
Recall that in [KRW] an example is given of a set B in a noetherian non-commutative graded structure, s.t. B is both a left standard set for $I_L(B)$ and a right standard set for $I_R(B)$, but is not a standard set for $I(B)$.

6 EFFECTIVE GRADED STRUCTURES

6.1 We say $\mathbf{A} := (A,\Gamma,v,G,in)$ is an *effective* graded structure iff:

E1) A,G,Γ are effective

E2) v,in are computable functions

E3) for each homogeneous element $g \in G$, it is possible to compute $a \in A$ s.t. $in(a) = g$, i.e. in^* is a computable function

E4) if $H \subset G-\{0\}$ is a finite set consisting of homogeneous elements, and h is a homogeneous element in the two-sided ideal generated by H, then it is possible to compute $l_1,\dots,l_s, r_1,\dots,r_s \in G$ homogeneous s.t. $\Sigma l_i \tilde{h}_i r_i = h$, with $h_i \in H$, and

$$deg(l_i) + deg(h_i) + deg(r_i) = deg(h).$$

E5) for each finite set $B \subset A-\{0\}$, $Syz(in(B))$ is finitely generated, and it is possible to compute explicitly a finite basis of it.

6.2 We say $\mathbf{A} := (A,\Gamma,v,G,in)$ is a *left-effective* graded structure iff E1, E2, E3 hold and, moreover:

E4L) if $h_1,\dots,h_s,h$ are non-zero homogeneous elements in G and h is in the left ideal generated by $\{h_1,\dots,h_s\}$, then it is possible to compute $l_1,\dots,l_s \in G$ homogeneous s.t. $\Sigma l_i h_i = h$, and $deg(l_i) + deg(h_i) = deg(h)$.

E5L) for each finite set $B \subset A-\{0\}$, $LSyz(in(B))$ is finitely generated, and it is possible to compute explicitly a finite basis of it.

6.3 The definitions are motivated by the fact that, if $\mathbf{A}$ is an effective graded structure, with G noetherian and $im(v)$ well-ordered, the following obvious generalizations of Buchberger's algorithms ([BUC1,2,3]; cf. also [B-R] for the commutative case), allow to compute standard bases for any two-sided ideal $I \subset A$.

We will make use of the following notations: if $B(1) \subset B(2) \subset \dots \subset B(t) \subset \dots$, $B(t) \subset A-\{0\}$ is a sequence of finite sets, we will denote:

$G(t)$ the free two-sided module over G with basis $\{e_b : b \in B(t)\}$, graded by assigning $deg(e_b) := deg(in(b)) = \upsilon(b)$;

$S_t : G(t) \to G$ the homogeneous morphism defined by
$S_t(\Sigma_{b\in B(t)} (\Sigma\, l_{ib}\, e_b\, r_{ib})) := \Sigma_{b\in B(t)} (\Sigma\, l_{ib}\, in(b)\, r_{ib})$.

With some abuse of notation, we will assume $G(t) \subset G(t+1)$, with the canonical inclusion, so that $Syz(in\{B(t)\}) \subset Syz(in\{B(t+1)\})$.

6.4 ALGORITHM

$g :=$ **Reduction**(**A**,B,f)

A := (A,Γ,υ,G,in) is an effective graded structure (with im(υ) well-ordered).

$B \subset A-\{0\}$ is a finite set.

$f \in A-\{0\}$.

$g \in A$ is s.t. $f - g$ has a Cauchy standard representation in terms of B, and, if $g \neq 0$, then $\upsilon(g) \leq \upsilon(f)$ and $in(g) \notin in(B)$.

    g := f
    While $g \neq 0$ **and** $in(g) \in (in(B))$ **do**
        compute (by E4) a homogeneous representation
            $in(g) = \Sigma_{b\in B(t)} (\Sigma\, l_{ib}\, in(b)\, r_{ib})$
        $g := g - \Sigma_{b\in B(t)} (\Sigma\, in^*(l_{ib})\, b\, in^*(r_{ib}))$

6.5 Correctness is an obvious consequence of I,2.5 and 2.6.
Termination is guaranteed only in case im(υ) is well-ordered, otherwise an infinite while-loop could occur (e.g. if B is a standard set and $f \in I\hat{}-I$).
If im(υ) is not well-ordered and B is a standard set, Procedure 6.4 is a semi-decision procedure for the problem "Is f not in I^?".

6.6 ALGORITHM

C := **StandardSet**(**A**,B)

A := (A,Γ,υ,G,in) is an effective graded structure with G noetherian and im(υ) well-ordered.

$B \subset A-\{0\}$ is a finite set.

C is a standard set of the two sided ideal generated by B.

    B(1) := B
    t := 1
    Compute (by E5) a finite homogeneous basis F_1 of $Syz(in\{B(1)\})$
    $F := F_1$
    While $F \neq \emptyset$ **do**

Choose $\sum_{b\in B(t)} (\sum l_{ib}\, e_b\, r_{ib}) \in F$

$F := F - \{\sum_{b\in B(t)} (\sum l_{ib}\, e_b\, r_{ib})\}$

$h := \sum_{b\in B(t)} (\sum \text{in}^*(l_{ib})\, b\, \text{in}^*(r_{ib}))$

$h :=$ **Reduction**$(A, B(t), h)$

If $h \neq 0$ **then**

$B(t+1) := B(t) \cup \{h\}$

Compute (by E4) a finite basis F_{t+1} of $\text{Syz}(\text{in}\{B(t+1)\})$ containing F_t

$F := F \cup (F_{t+1} - F_t)$

$t := t+1$

$C := B(t)$

6.7 If im(v) is well-ordered, the calls of the procedure **Reduction** halt; moreover, since $\text{in}(B(t)) \subsetneq \text{in}(B(t+1))$, noetherianity of G guarantees that, after a finite number of computations, one reaches $F = \emptyset$.

Then, the basis $B(t)$ of I satisfies A9) since F_t is a homogeneous basis of $\text{Syz}(\text{in}\{B(t)\})$ and for each $f \in F_t$, $s(\text{in}^*(f))$ has a Cauchy standard representation in terms of $B(t)$.

PART III
REPRESENTATIONS

INTRODUCTION

In most instantiations of graded structures, G is a subring of A, so all computations can be done in A; this doesn't happen in the general case (Ex. I.1.8 and Ex. I.1.10 are suitable examples).
Such a restriction could seem to be important in order to derive a "division theorem", i.e. results on canonical representatives in A modulo an ideal by means of standard sets, which generalise the results given by Buchberger [BUC1,2] for polynomial rings and by Galligo [GAL1] for formal power series rings.
In order to show that this is not the case, we intend to show here that there is still a strict relationship between elements in G^ and in A^, so that any of the two sets can be used to represent the other one.
Inspired both from Kandri Rody and Weispfenning presentation of algebras of solvable type [KRW], and by the identification (usual in Computer Algebra) of integers and polynomials over $\mathbb{Z}_p$, we show the following:

> there is a *set* isomorphism between A^ and G^; such isomorphism is *not*, in general, a ring isomorphism, but, under it, G^ inherits a second ring structure; the "new" sum and product differ from the original ones, because of the presence of "carries" of lesser degree.

Because of this result, assuming that G has canonical representatives for homogeneous elements modulo a homogeneous ideal, we are able to show that the same is true for A^ modulo any ideal, so to give a "division theorem" generalising both Buchberger's and Galligo's results.
In general, however (see the remark below), such canonical representatives can be computed only if A is noetherian and im(v) well-ordered, so we postpone the algorithmic discussion to Part IV.
This result has another consequence related to the existence of a standard set algorithm in the general case: if A is the polynomial ring graded over a negative semigroup Γ, and **A** denotes the associated graded structure, then A^ is the formal power series ring, a ring which is not effective, on which a standard set algorithm is still lacking (also under suitable effectiveness conditions on the elements involved), and whose division theorem allows just the computation of "truncated" canonical representations.
Because of the result above, one can conclude that this is the general situation, since the elements of G^ are infinite formal sums of homogeneous elements of G, i.e. G^ is truly a "ring of series".
This obviously leads to a pessimistic point of view about the existence

of a general standard set algorithm.
After describing a suitable ring structure on G^, different from the canonical one, which we call *twisted completion* and which is inspired by the presentation of algebras of solvable type in [KRW] (§1), we show that a set isomorphism exists between A^ and G^ and that under this isomorphism, G^ inherits a twisted completion ring structure (§2); we are then able to state a general division theorem in A^ (§3).

1 TWISTED COMPLETIONS OF A GRADED RING

1.1 Let G be a Γ-graded ring, with Γ inf-limited.
Let us denote **G** := (G,Γ,deg,G,M) the graded structure defined on G as in I.1.5 and **G^** := (G^,Γ,deg^,G,M^) its completion.
If $g \in G^\wedge - \{0\}$, define $R(g) := g - M^\wedge(g)$; remark that either $R(g) = 0$ or $\deg^\wedge(R(g)) < \deg^\wedge(g)$.

1.2 If $g \in G^\wedge$, there are unique $h_0,\dots,h_n,\dots \in G$ s.t.

h_n is homogeneous

if $h_n = 0$ then $h_{n+1} = 0$

if $h_n \neq 0$ then $\deg(h_n) < \deg(h_{n-1})$

$g = \lim \Sigma_{i=1\dots n} h_i$

With some abuse of notation we will write $g = \Sigma_{i=0\dots\infty} h_i$.
Also G^ is isomorphic to the ring of all functions $g : \Gamma \to G$ s.t.

i) for all $\gamma \in \Gamma$, $g(\gamma) \in G(\gamma)$

ii) for all sequences $\gamma_1 < \gamma_2 < \dots < \gamma_n < \dots$ there is N s.t. $g(\gamma_n) = 0$ if $n \geq N$

1.3 LEMMA If J is a homogeneous ideal of G, then $M(J) = J$ and $J^\wedge = J\,G^\wedge$.
Also, if B is a basis of J consisting of homogeneous elements, then B is a standard set and a standard basis for J in **G** and for J^ in **G^**.

1.4 DEFINITION Let $\oplus, \otimes$ be two binary operations on G^ s.t.:

i) $(G^\wedge, \oplus, \otimes)$ is a ring

ii) the null element and the unity of $(G^\wedge, \oplus, \otimes)$ coincide with those of G^

iii) for each $g \in G^\wedge$, $g = M^\wedge(g) \oplus R(g)$

iv) for each g',g" homogeneous elements in G s.t. $\deg(g') > \deg(g")$

$$g' \oplus g" = g' + g"$$

v) for each g',g" homogeneous elements in G s.t. $\deg(g') = \deg(g")$, there is $\mathbf{AC}(g',g") \in G^\wedge$:

$$g' \oplus g" = g' + g" + \mathbf{AC}(g',g")$$

$$\deg^\wedge(\mathbf{AC}(g',g")) < \deg(g')$$

vi) for each g',g" homogeneous elements in G there is $\mathbf{MC}(g',g") \in G^\wedge$: